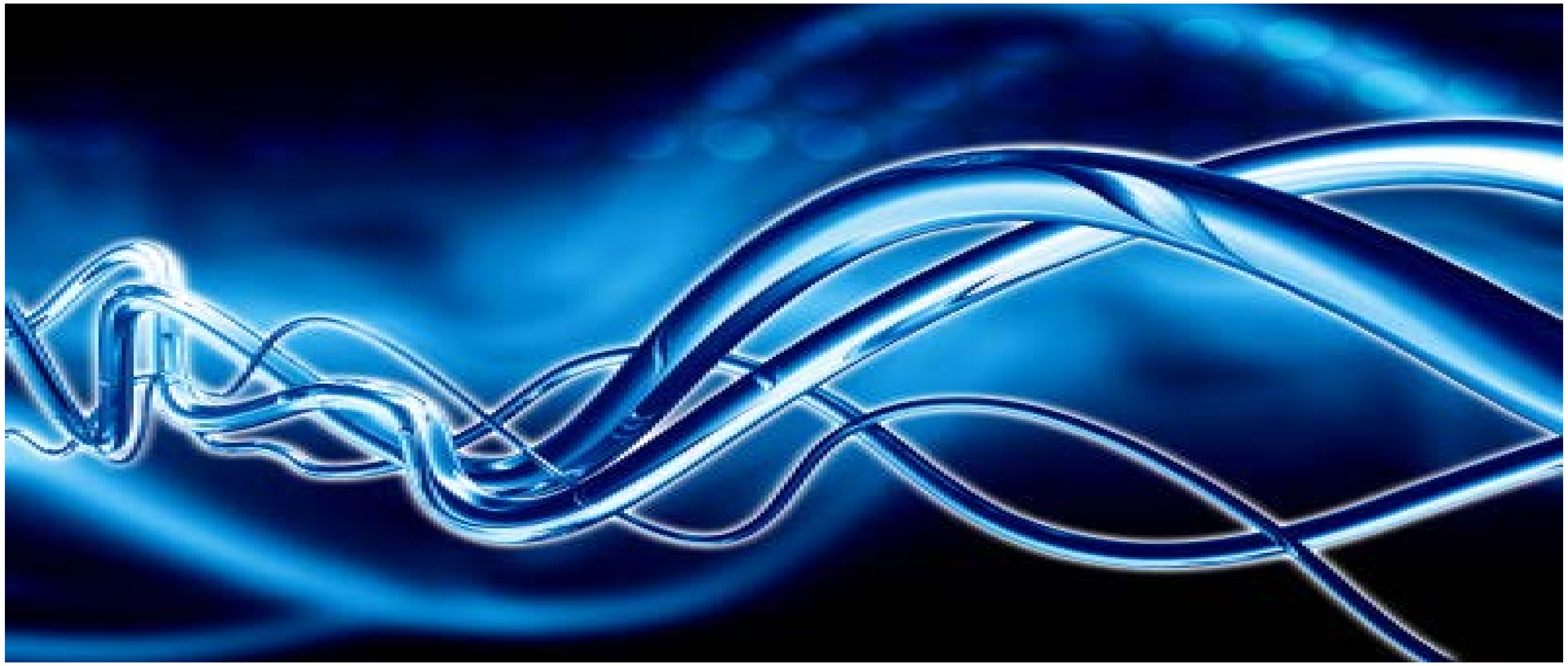
Signals and Systems



3. The Laplace Transform Part 1

The Laplace Transform

- **Contents**
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  Properties of the two-sided Laplace transform
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  Properties of the one-sided Laplace transform
  Circuit theory revisited

- **Book:**
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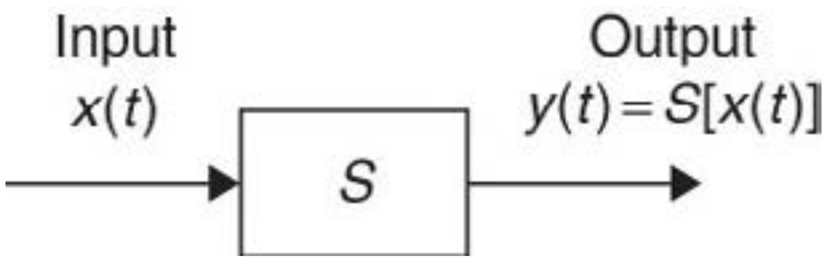
The Laplace Transform

- Given an LTI system with a single input and a single output
- Input signal: $x(t)$
- Output signal: $y(t)$
- We have seen that the output signal is given by the convolution of the input signal $x(t)$ and the impulse response $h(t)$:

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau = \int_{\tau=-\infty}^{\infty} x(t-\tau)h(\tau)\,\mathrm{d}\tau$$



The Laplace Transform

- Let the input signal be given by

$$x(t) = e^{st} \quad \text{with} \quad s \in \mathbb{C}$$

- The corresponding output signal is

$$y(t) = \int_{\tau=-\infty}^{\infty} e^{s(t-\tau)} h(\tau)\, \mathrm{d}\tau = \int_{\tau=-\infty}^{\infty} h(\tau) e^{-s\tau}\, \mathrm{d}\tau\, e^{st} = H(s)x(t)$$

with

$$H(s) = \int_{\tau=-\infty}^{\infty} h(\tau) e^{-s\tau}\, \mathrm{d}\tau$$

The Laplace Transform

- We observe that the output signal is a multiple of the input signal (provided the integral converges, of course): $y(t) = H(s)e^{st}$

$$\underbrace{S}_{A}\{\underbrace{e^{st}}_{u}\} = \underbrace{H(s)}_{\lambda}\underbrace{e^{st}}_{u}$$

  Compare this with a standard eigenvalue problem in linear algebra:

$$Au = \lambda u$$

- $H(s)$ is an *eigenvalue* of the LTI system corresponding to the *eigenfunction* $x(t) = e^{st}$

The Laplace Transform

- The expression for $H(s)$ is precisely the definition of the two-sided Laplace transform of $h(t)$
- Two-sided Laplace transform of a signal $x(t)$:

$$X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t$$

defined, of course, for those $s \in \mathbb{C}$ for which the integral converges

The Laplace Transform

- To investigate under what condition(s) convergence takes place, we consider
    * the Laplace transform of causal signals: $x(t) = 0$ for $t < 0$
    * the Laplace transform of anti-causal signals: $x(t) = 0$ for $t > 0$
    * the Laplace transform of noncausal signals

The Laplace Transform

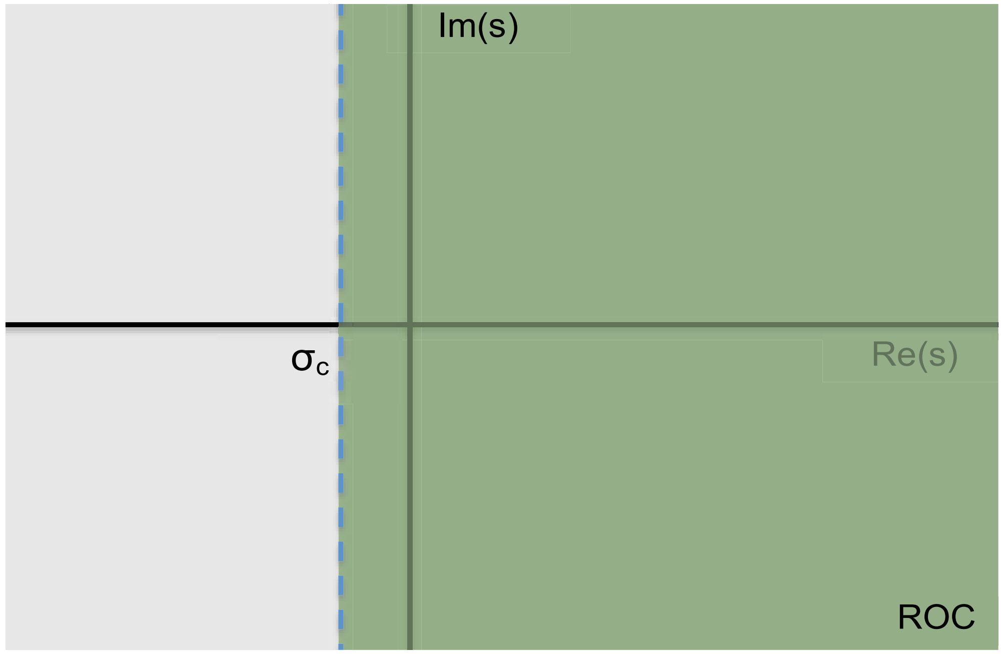
- A causal signal $x(t)$ is said to be of *exponential order* if there exists constants A and α such that

$$|x(t)| \leq Ae^{\alpha t} \quad \text{for } t \geq 0$$

- It can be shown that for causal signals of exponential order there exists a unique number $-\infty \leq \sigma_c < \infty$ such that the Laplace integral converges for $\mathrm{Re}(s) > \sigma_c$
- The number σ_c is called the *abscissa of convergence*

The Laplace Transform

- The set $\{s \in \mathbb{C}; \text{Re}(s) > \sigma_c\}$ is called the *Region of Convergence* (ROC)
- To avoid confusion, we sometimes write ROC_x to indicate the ROC of the Laplace transform of a signal $x(t)$
- Note that the ROC of a causal signal (of exponential order) is a *right-half plane* (unless $\sigma_c = -\infty$, of course)
- It can also be shown that $X(s)$ is *analytic* on its ROC



The Laplace Transform

- **Example 1:** The two-sided Laplace transform of the Heaviside unit step function $u(t)$

$$U(s) = \int_{t=-\infty}^{\infty} u(t) e^{-st} \, \mathrm{d}t = \int_{t=0}^{\infty} e^{-st} \, \mathrm{d}t = \frac{1}{s} \quad \text{for } \mathrm{Re}(s) > 0$$

- In this case $\mathrm{ROC} = \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$

The Laplace Transform

- **Example 2:** The two-sided Laplace transform of a scaled rectangular pulse function $x(t) = p\left(\frac{t}{T}\right)$, $T > 0$

$$X(s) = \int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t = \int_{t=0}^{T} e^{-st}\,\mathrm{d}t = \frac{1}{s}\left(1 - e^{-sT}\right), \qquad s \in \mathbb{C}$$

- Note that there is no pole at $s = 0$
- In this case ROC $= \mathbb{C}$

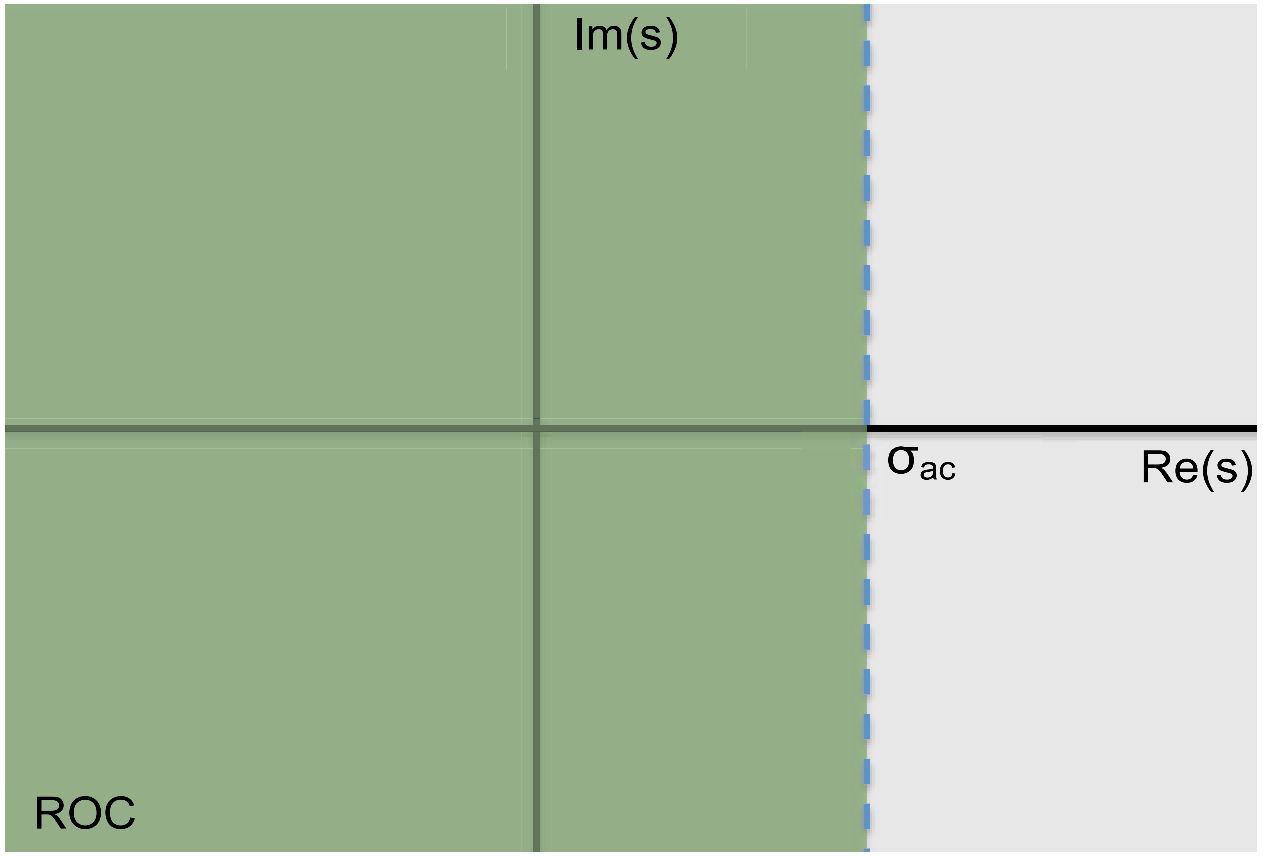
The Laplace Transform

- An anti-causal signal $x(t)$ is said to be of exponential order if there exists constants B and β such that

$$|x(t)| \leq Be^{\beta t} \quad \text{for } t \leq 0$$

- For anti-causal signals of exponential order it can be shown that there exists a number $-\infty < \sigma_{\text{ac}} \leq \infty$ such that the Laplace integral converges for $\text{Re}(s) < \sigma_{\text{ac}}$
- The number σ_{ac} is again called the abscissa of convergence and the ROC is a *left-half plane* (unless $\sigma_{\text{ac}} = \infty$)
- The Laplace transform is analytic on the ROC

The Laplace Transform



The Laplace Transform

- **Example.** The two-sided Laplace transform of $x(t) = u(-t)$

$$X(s) = \int_{t=-\infty}^{\infty} x(t) e^{-st} \, \mathrm{d}t = \int_{t=-\infty}^{0} e^{-st} \, \mathrm{d}t = -\frac{1}{s} \quad \text{for } \mathrm{Re}(s) < 0$$

- For this signal the ROC $= \{s \in \mathbb{C}; \mathrm{Re}(s) < 0\}$

- *Specifying the ROC is important!*
- For example, $X(s) = 1/s$ can be
    * the Laplace transform of the causal signal $x(t) = u(t)$, or
    * the Laplace transform of the anti-causal signal $x(t) = -u(-t)$
- Which one is intended becomes clear by specifying the ROC

The Laplace Transform

- Finally, what about noncausal signals?
- For such a signal we write

$$x(t) = x(t) \cdot 1 = x(t)\big[u(t) + u(-t)\big] = x_{\mathrm{c}}(t) + x_{\mathrm{ac}}(t)$$

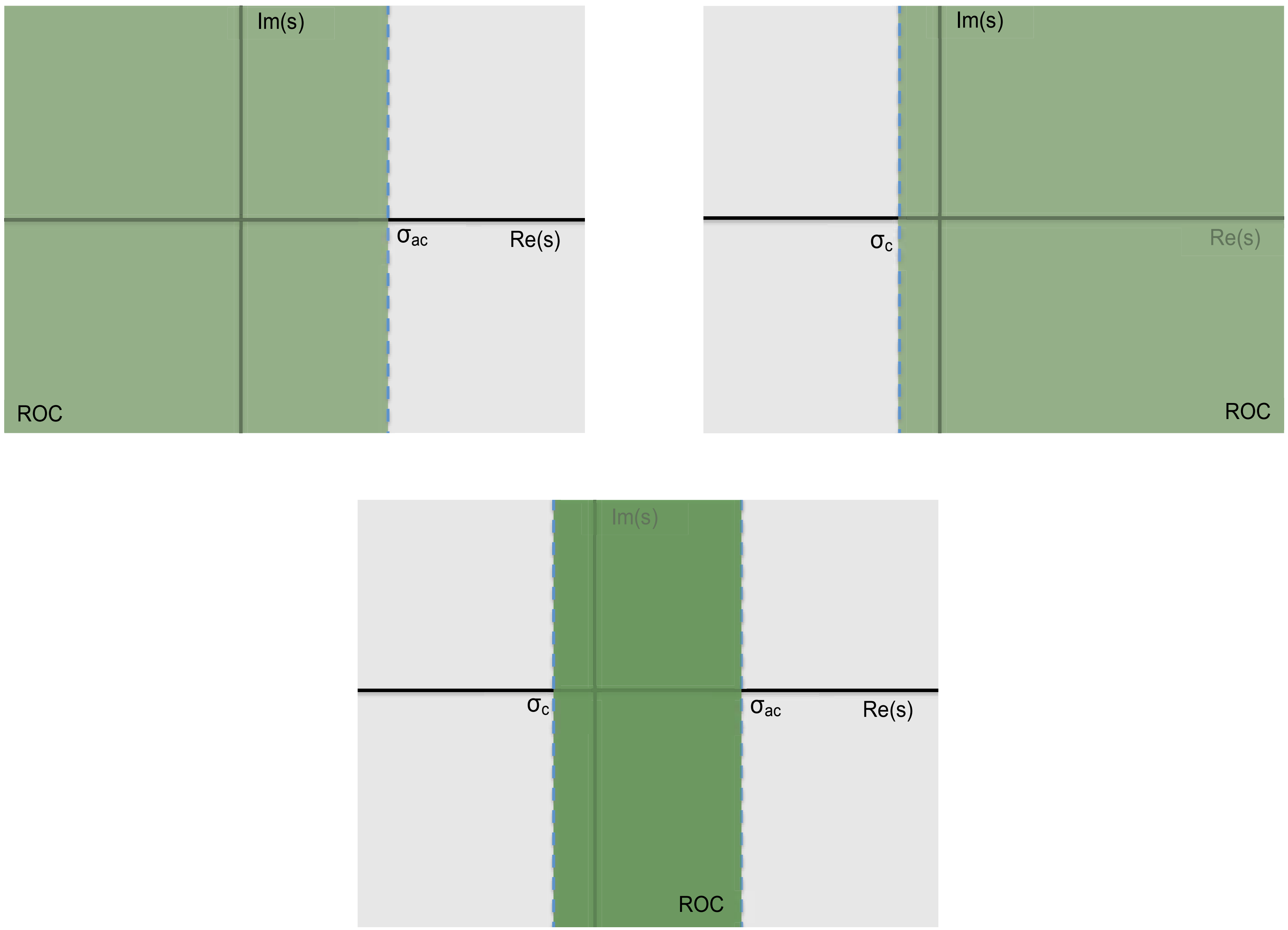
with

$$x_{\mathrm{c}}(t) = x(t)u(t) \quad \text{and} \quad x_{\mathrm{ac}}(t) = x(t)u(-t)$$

The Laplace Transform

- The Laplace transform of $x_c(t)$ is $X_c(s)$ with a region of convergence given by ROC_{x_c}
- The Laplace transform of $x_{ac}(t)$ is $X_{ac}(s)$ with a region of convergence given by $\text{ROC}_{x_{ac}}$
- The Laplace transform of the noncausal signal $x(t)$ is given by $X(s) = X_c(s) + X_{ac}(s)$ with a region of convergence given by $\text{ROC}_x = \text{ROC}_{x_c} \cap \text{ROC}_{x_{ac}}$
- The two-sided Laplace transform of $x(t)$ does not exist if $\text{ROC}_x = \emptyset$
- If the two-sided Laplace transform exists then in general its ROC is a *strip* in the complex s-plane

The Laplace Transform



- **Example 1:** The two-sided Laplace transform of the noncausal signal $x(t) = e^{-|t|}$ is

$$X(s) = -\frac{2}{s^2 - 1} \quad \text{with} \quad -1 < \mathrm{Re}(s) < 1$$

- **Example 2:** The two-sided Laplace transform of the noncausal signal $x(t) = e^{at}$ with $a \in \mathbb{C}$ does not exist

The Laplace Transform

- The two-sided Laplace transform of the Dirac distribution is

$$\int_{t=-\infty}^{\infty} \delta(t) e^{-st}\, \mathrm{d}t = 1, \qquad s \in \mathbb{C}$$

- The two-sided Laplace transform of the derivative of the Dirac distribution is

$$\int_{t=-\infty}^{\infty} \delta'(t) e^{-st}\, \mathrm{d}t$$

- Recall that

$$f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$$

The Laplace Transform

- Using the above equation with $f(t) = e^{-st}$ gives

$$e^{-st}\delta'(t) = s\delta(t) + \delta'(t)$$

- and substitution results in

$$\int_{t=-\infty}^{\infty} \delta'(t)e^{-st}\,\mathrm{d}t = s, \qquad s \in \mathbb{C}$$

since

$$\int_{t=-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1 \quad \text{and} \quad \int_{t=-\infty}^{\infty} \delta'(t)\,\mathrm{d}t = 0$$

The Laplace Transform

Two-Sided Laplace Transforms

Time signal	**Two-sided Laplace transform**	**ROC**	**parameters**
$e^{at}u(t)$	$\frac{1}{s-a}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$	$a \in \mathbb{C}$
$-e^{at}u(-t)$	$\frac{1}{s-a}$	$\mathrm{Re}(s) < \mathrm{Re}(a)$	$a \in \mathbb{C}$
$\frac{t^{k-1}e^{at}}{(k-1)!}u(t)$	$\frac{1}{(s-a)^k}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$	$a \in \mathbb{C}$, $k \in \mathbb{N}$
$-\frac{t^{k-1}e^{at}}{(k-1)!}u(-t)$	$\frac{1}{(s-a)^k}$	$\mathrm{Re}(s) < \mathrm{Re}(a)$	$a \in \mathbb{C}$, $k \in \mathbb{N}$
$e^{at}\cos(\Omega_0 t)u(t)$	$\frac{s-a}{(s-a)^2+\Omega_0^2}$	$\mathrm{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$e^{at}\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{(s-a)^2+\Omega_0^2}$	$\mathrm{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$\delta(t)$	1	$\mathbb{C}$	–
$\delta'(t)$	s	$\mathbb{C}$	–

- Note that the Laplace transforms of the signals in the above table are *rational functions*

$$X(s) = \frac{\text{some polynomials in } s}{\text{some other polynomial in } s}$$

The Laplace Transform

- **Convolution** Let $y(t) = x(t) * h(t)$
- $X(s)$ is the two-sided Laplace transform of $x(t)$ with a region of convergence ROC_x
- $H(s)$ is the two-sided Laplace transform of $h(t)$ with a region of convergence ROC_h
- The Laplace transform of $y(t)$ is

$$Y(s) = X(s)H(s) \quad \text{with } \text{ROC}_y = \text{ROC}_x \cap \text{ROC}_h$$

The Laplace Transform

- A convolution product in the time-domain is transformed into an ordinary product in the s-domain!
- Let's verify this statement

The Laplace Transform

- We start with the definition of the two-sided Laplace transform

$$Y(s) = \int_{t=-\infty}^{\infty} y(t) e^{-st} \, \mathrm{d}t$$

- Substitution of the convolution integral gives

$$Y(s) = \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau) \, \mathrm{d}\tau \, e^{-st} \, \mathrm{d}t$$

The Laplace Transform

- Interchanging the order of integration, we get

$$
\begin{aligned}
Y(s) &= \int_{\tau=-\infty}^{\infty} x(\tau) \int_{t=-\infty}^{\infty} h(t-\tau) e^{-st}\, \mathrm{d}t\, \mathrm{d}\tau \\
&\stackrel{p=t-\tau}{=} \int_{\tau=-\infty}^{\infty} x(\tau) \int_{p=-\infty}^{\infty} h(p) e^{-s(p+\tau)}\, \mathrm{d}p\, \mathrm{d}\tau \\
&= \int_{\tau=-\infty}^{\infty} x(\tau) e^{-s\tau}\, \mathrm{d}\tau \int_{p=-\infty}^{\infty} h(p) e^{-sp}\, \mathrm{d}p \\
&= X(s) H(s)
\end{aligned}
$$

with $s \in \mathrm{ROC}_x \cap \mathrm{ROC}_h$

The Laplace Transform

- Given an LTI system with input signal $x(t)$ and output signal $y(t)$
- Let $h(t)$ denote the impulse response of this system
- Output signal:

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau$$

- In the Laplace- or s-domain:

$$Y(s) = H(s)X(s) \qquad \text{with } s \in \mathrm{ROC}_x \cap \mathrm{ROC}_h$$

- $H(s)$ is called the *transfer function* of the system

- **Example** Let $x(t) = u(t)$ and $h(t) = e^{-t}u(t)$. We are interested in the convolution product $y(t) = x(t) * h(t)$
- Computing this product directly, we find

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ \int_{\tau=0}^{t} h(t-\tau)\,\mathrm{d}\tau = 1 - e^{-t} & \text{for } t > 0 \end{cases}$$

- The two-sided Laplace transform of $x(t)$ is given by

$$X(s) = \frac{1}{s}, \qquad \mathrm{Re}(s) > 0$$

The Laplace Transform

- The two-sided Laplace transform of $h(t)$ is given by

$$H(s) = \frac{1}{s+1}, \qquad \mathrm{Re}(s) > -1$$

- The two-sided Laplace transform of $y(t)$ is given by

$$Y(s) = X(s)H(s) = \frac{1}{s(s+1)}, \qquad \mathrm{Re}(s) > 0$$

- What time signal corresponds this Laplace domain function?
- Observe that

$$Y(s) = \frac{1}{s} - \frac{1}{s+1}, \qquad \mathrm{Re}(s) > 0$$

- Using the table, we find $y(t) = (1 - e^{-t})u(t)$ or

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 - e^{-t} & \text{for } t > 0 \end{cases}$$

- Much more on the inverse Laplace transform in the next lecture

- **Differentiation in the Laplace-domain**
- Let $X(s)$ be the two-sided Laplace transform of a signal $x(t)$ with a region of convergence given by ROC_x
- We have stated that $X(s)$ is analytic on ROC_x
- The Laplace transform can therefore be differentiated
- We have

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Interchanging the order of differentiation and integration, we find

$$\frac{\mathrm{d}X(s)}{\mathrm{d}s} = \int_{t=-\infty}^{\infty} [-tx(t)]e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

The Laplace Transform

- The expression on the right is the Laplace transform of $-tx(t)$

- We conclude that

$$-tx(t) \text{ transforms into } \frac{\mathrm{d}X(s)}{\mathrm{d}s} \qquad s \in \mathrm{ROC}_x$$

- Differentiation in the Laplace-domain corresponds to multiplication by $-t$ in the time-domain

- **Differentiation in the time-domain**
- Suppose we are given a time-domain signal $x(t)$ with a two-sided Laplace transform $X(s)$, $s \in \mathrm{ROC}_x$
- What is the Laplace transform of

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}?$$

The Laplace Transform

- By definition, we have

$$\begin{aligned} Y(s) &= \int_{t=-\infty}^{\infty} \frac{\mathrm{d}x(t)}{\mathrm{d}t} e^{-st}\,\mathrm{d}t \\ &= \lim_{T\to\infty} x(t)e^{-st}\Big|_{t=-T}^{T} - \int_{t=-\infty}^{\infty} x(t)\left(-se^{-st}\right)\mathrm{d}t \\ &= s\int_{t=-\infty}^{\infty} x(t)e^{-st}\,\mathrm{d}t \\ &= sX(s), \qquad \text{with } s \in \mathrm{ROC}_x \end{aligned}$$

- The first term on the right-hand side on the second line (blue formula) vanishes because we assumed that $X(s)$ exists

- We have found that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \quad \text{transforms into } sX(s) \qquad s \in \mathrm{ROC}_x$$

The Laplace Transform

- Differentiation in the time-domain transforms into multiplication by s in the Laplace domain!

- **Integration in the time-domain**
- Suppose that we are again given a time-domain signal $x(t)$ with a two-sided Laplace transform $X(s)$, $s \in \mathrm{ROC}_x$
- What is the Laplace transform of

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)\,\mathrm{d}\tau\,?$$

The Laplace Transform

- Observe that y is the convolution of x and the Heaviside unit step function u:

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,u(t-\tau)\,\mathrm{d}\tau$$

- Using the convolution property, we find

$$Y(s) = X(s) \cdot \frac{1}{s}, \qquad s \in \mathrm{ROC}_x \cap \mathrm{ROC}_u$$

- If $\mathrm{ROC}_x \cap \mathrm{ROC}_u = \emptyset$ then the Laplace transform of $y(t)$ does not exist

The Laplace Transform

- Since $\mathrm{ROC}_u = \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$, we can also write

$$Y(s) = \frac{1}{s} X(s), \qquad s \in \left\{\mathrm{ROC}_x | \mathrm{Re}(s) > 0\right\}$$

- We have found that

$$\int_{\tau=-\infty}^{t} x(\tau)\,\mathrm{d}\tau \quad \text{transforms into } \frac{1}{s} X(s) \qquad s \in \left\{\mathrm{ROC}_x | \mathrm{Re}(s) > 0\right\}$$

- Integration in the time-domain transforms in to division by s in the Laplace domain!

- **Shift in the time-domain**
- Again we have a signal $x(t)$ with a two-sided Laplace transform $X(s)$, $s \in \mathrm{ROC}_x$
- Let $y(t)$ be a shifted version of $x(t)$ with time shift $\tau \in \mathbb{R}$:

$$y(t) = x(t+\tau), \qquad \tau \in \mathbb{R}$$

The Laplace Transform

- What is the Laplace transform of $y(t)$?
- We compute

$$
\begin{aligned}
Y(s) &= \int_{t=-\infty}^{\infty} x(t+\tau)e^{-st}\,\mathrm{d}t \\
&\overset{p=t+\tau}{=} \int_{p=-\infty}^{\infty} x(p)e^{-s(p-\tau)}\,\mathrm{d}p \\
&= e^{s\tau}\int_{p=-\infty}^{\infty} x(p)e^{-sp}\,\mathrm{d}p \\
&= e^{s\tau}X(s), \qquad s \in \mathrm{ROC}_x
\end{aligned}
$$

- We have found that

$$x(t+\tau) \text{ transforms into } e^{s\tau}X(s), \qquad s \in \mathrm{ROC}_x$$

- **Example.** Suppose the two-sided Laplace transform of a signal $h(t)$ is given by

$$H(s) = \frac{1}{1-e^{-sT}} \qquad \text{with } T > 0 \quad \text{and} \quad \mathrm{ROC}_h = \{s \in \mathbb{C}; \mathrm{Re}(s) > 0\}$$

- What is $h(t)$?

- Set $z = e^{-sT}$. We then have

$$\frac{1}{1-e^{-sT}} = \frac{1}{1-z}$$

- Now recall the power series

$$\frac{1}{1-z} = 1 + z + z^2 + \ldots \qquad \text{for } |z| < 1 \text{ and } z \in \mathbb{C}$$

- In our case, we have with $s = \sigma + \mathrm{j}\Omega$:

$$|z| = \left|e^{-sT}\right| = \left|e^{-\sigma T - \mathrm{j}\Omega T}\right| = \left|e^{-\sigma T}\right| \cdot \left|e^{-\mathrm{j}\Omega T}\right| = e^{-\sigma T}, \qquad \text{since } \left|e^{-\mathrm{j}\Omega T}\right| = 1$$

- We also have $\mathrm{Re}(s) = \sigma > 0$ and $T > 0$. Consequently,

$$|z| = e^{-\sigma T} < 1$$

and

$$\frac{1}{1 - e^{-sT}} = 1 + e^{-sT} + e^{-2sT} + \ldots$$

The Laplace Transform

- Using the Laplace transform of the Dirac distribution and the shifting property, we find

$$h(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \ldots$$

- Suppose $x(t)$ is a causal signal with support $(0, T_x)$
- For example, $x(t) = \Lambda(t)$, support $(0, 2)$
- The Laplace transform of $x(t)$ is $X(s)$, $s \in \mathbb{C}$

The Laplace Transform

- Given now an LTI system with a transfer function

$$H(s) = \frac{1}{1 - e^{-sT}} \qquad \text{with } T > T_x \text{ and } \mathrm{Re}(s) > 0$$

- The Laplace transform of the output signal is

$$Y(s) = \frac{X(s)}{1 - e^{-sT}}, \qquad \mathrm{Re}(s) > 0$$

The Laplace Transform

- The corresponding output signal is given by the convolution integral

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)\,x(t-\tau)\,\mathrm{d}\tau$$

with

$$h(t) = \delta(t) + \delta(t-T) + \delta(t-2T) + \ldots = \sum_{k=0}^{\infty} \delta(t-kT)$$

The Laplace Transform

- Substitution gives

$$y(t) = \int_{\tau=-\infty}^{\infty} h(\tau)x(t-\tau)\,\mathrm{d}\tau = \int_{\tau=-\infty}^{\infty} \sum_{k=0}^{\infty} \delta(\tau - kT)\, x(t-\tau)\,\mathrm{d}\tau$$

$$= \sum_{k=0}^{\infty} \int_{\tau=-\infty}^{\infty} \delta(\tau - kT) x(t-\tau)\,\mathrm{d}\tau = \sum_{k=0}^{\infty} x(t-kT)$$

$$= x(t) + x(t-T) + x(t-2T) + \ldots$$

- We have constructed a periodic extension of $x(t)$ for $t > 0$

- **Shift in the Laplace domain**
- Let $X(s)$ be the two-sided Laplace transform of $x(t)$ with $s \in \text{ROC}_x$
- Is there a time-domain signal that corresponds to $X(s-a)$ with $s-a \in \text{ROC}_x$?
- We use the definition of the Laplace transform

$$X(s-a) = \int_{t=-\infty}^{\infty} x(t)\, e^{-(s-a)\,t}\, \mathrm{d}t = \int_{t=-\infty}^{\infty} e^{at}\, x(t)\, e^{-st}\, \mathrm{d}t$$

- The answer is yes

$$e^{at}\, x(t) \text{ transforms into } X(s-a), \qquad s-a \in \text{ROC}_x$$

- **Scaling**
- Let $x(t)$ have a two-sided Laplace transform $X(s)$ with $s \in \mathrm{ROC}_x$
- Given a nonzero real number a, what is the Laplace transform of

$$y(t) = x(at)\,?$$

- We use the definition of the Laplace transform

The Laplace Transform

- For $a > 0$, we find

$$
\begin{aligned}
Y(s) &= \int_{t=-\infty}^{\infty} y(t)e^{-st}\,\mathrm{d}t = \int_{t=-\infty}^{\infty} x(at)e^{-st}\,\mathrm{d}t \\
&\overset{\tau=at}{=} \frac{1}{a}\int_{\tau=-\infty}^{\infty} x(\tau)e^{-(s/a)\tau}\,\mathrm{d}\tau \\
&= \frac{1}{a}X\left(\frac{s}{a}\right), \qquad s/a \in \mathrm{ROC}_x
\end{aligned}
$$

The Laplace Transform

- Similarly, for $a < 0$ we obtain

$$Y(s) = -\frac{1}{a} X\left(\frac{s}{a}\right) \qquad s/a \in \mathrm{ROC}_x$$

- Combining both results, we have

$$x(at) \text{ transforms into } \frac{1}{|a|} X(\frac{s}{a}) \qquad \text{for } a \in \mathbb{R} \setminus \{0\} \quad \text{and with} \quad \frac{s}{a} \in \mathrm{ROC}_x$$

- **Example**

- *Switch on.* We have seen that the two-sided Laplace transform of the Heaviside unit step function $u(t)$ is given by

$$U(s) = \frac{1}{s} \quad \text{with} \quad s \in \text{ROC}_u = \{s \in \mathbb{C}; \text{Re}(s) > 0\}$$

- *Switch off.* We have also seen that the two-sided Laplace transform of the anti-causal switch-off signal $f(t) = u(-t)$ is

$$F(s) = -\frac{1}{s} \quad \text{with} \quad s \in \text{ROC}_f = \{s \in \mathbb{C}; \text{Re}(s) < 0\}$$

- Clearly,

$$F(s) = U(-s) \quad \text{with } s \in \text{ROC}_f \text{ or } -s \in \text{ROC}_u$$

The Laplace Transform

Properties of the Two-Sided Laplace Transform

Property	**Time signal**	**Two-sided Laplace transform**	**ROC**	**Parameters**
Convolution	$y(t) = h(t) * x(t)$	$Y(s) = H(s)X(s)$	$\mathrm{ROC}_h \cap \mathrm{ROC}_x$	–
Diff. s-domain	$-tx(t)$	$\frac{\mathrm{d}X(s)}{\mathrm{d}s}$	ROC_x	–
Diff. t-domain	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$sX(s)$	ROC_x	–
Int. t-domain	$\int_{\tau=-\infty}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s}X(s)$	$\{\mathrm{ROC}_x \mid \mathrm{Re}(s) > 0\}$	–
Shift t-domain	$x(t+\tau)$	$e^{s\tau}X(s)$	ROC_x	$\tau \in \mathbb{R}$
Shift s-domain	$e^{at}x(t)$	$X(s-a)$	$s-a \in \mathrm{ROC}_x$	$a \in \mathbb{C}$
Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{s}{a}\right)$	$s/a \in \mathrm{ROC}_x$	$a \in \mathbb{R} \setminus \{0\}$

- **The One-Sided Laplace Transform**
- Let $x(t)$ denote a causal signal: $x(t) = x(t)u(t)$
- The two-sided Laplace transform simplifies to

$$X(s) = \int_{t=0}^{\infty} x(t) e^{-st} \, \mathrm{d}t \qquad s \in \mathrm{ROC}_x$$

- This transform is known as the *one-sided Laplace transform*

The Laplace Transform

- A separate study is warranted, since many (most/all) signals and systems encountered in practice/Nature are causal
- Switch-on phenomena (initial-value problems) are conveniently studied using the one-sided Laplace transform
- To incorporate the Dirac distribution $\delta(t)$, we define the one-sided Laplace transform of a signal $x(t)$ as

$$X(s) = \int_{t=0^-}^{\infty} x(t) e^{-st} \, \mathrm{d}t = \lim_{\epsilon \downarrow 0} \int_{t=-\epsilon}^{\infty} x(t) e^{-st} \, \mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

The Laplace Transform

- Many properties of the two-sided Laplace transform carry over to the one-sided Laplace transform
- We only discuss three properties of the one-sided transform that do not have a two-sided counterpart

- **Differentiation in the time-domain**
- Let $X(s)$ denote the one-sided Laplace transform of the time-domain signal $x(t)$
- What is the one-sided Laplace transform of

$$y(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}\,?$$

- By definition

$$Y(s) = \int_{t=0^-}^{\infty} \frac{\mathrm{d}x(t)}{\mathrm{d}t} e^{-st} \mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Integration by parts gives

$$Y(s) = \lim_{T\to\infty, \epsilon\downarrow 0} x(t)e^{-st}|_{-\epsilon}^{T} - \int_{t=0^-}^{\infty} x(t)[-se^{-st}]\,\mathrm{d}t = -x(0^-) + sX(s)$$

with $x(0^-) = \lim_{\epsilon\downarrow 0} x(-\epsilon)$ and $s \in \mathrm{ROC}_x$

- We have found that

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} \text{ transforms into } sX(s) - x(0^-) \qquad s \in \mathrm{ROC}_x$$

- Similarly, by repeated integration by parts we find that

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} \text{ transforms into } s^2 X(s) - s x(0^-) - \frac{\mathrm{d}x(t)}{\mathrm{d}t}\Big|_{t=0^-} \qquad s \in \mathrm{ROC}_x$$

- **Abel's initial-value theorem**
- Let $X(s)$ be the one-sided Laplace transform of $x(t)$, $s \in \mathrm{ROC}_x$
- Abel's initial-value theorem states that

$$\lim_{s \to \infty} sX(s) = x(0^+),$$

with $x(0^+) = \lim_{\epsilon \downarrow 0} x(\epsilon)$, provided $x(t)$ is regular at $t = 0$

- Left-hand side: Laplace-domain
- Right-hand side: time-domain

- We do not prove this theorem, we only make it plausible
- Consider

$$sX(s) = \int_{t=0^-}^{\infty} x(t)\, s e^{-st}\,\mathrm{d}t, \qquad s \in \mathrm{ROC}_x$$

- Recall that the ROC is some right-half plane (or $\mathbb{C}$)
- Take s real, positive, and sufficiently large so that $s \in \mathrm{ROC}_x$
- For increasing values of s, the function

se^{-st} behaves as a Dirac distribution!

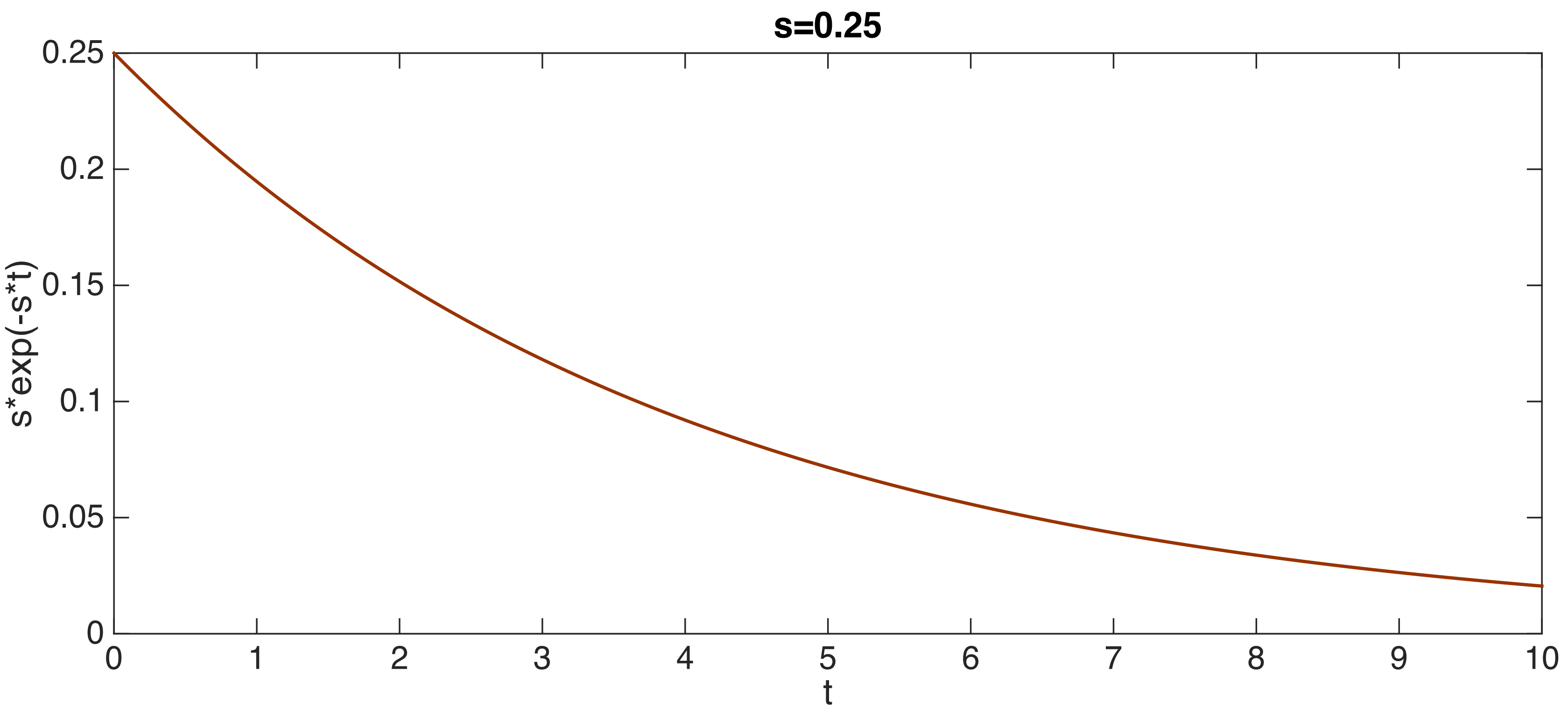
- For example, consider a regular causal signal with an abscissa of convergence $\sigma_{\mathrm{c}} = 0$

- In addition, we have

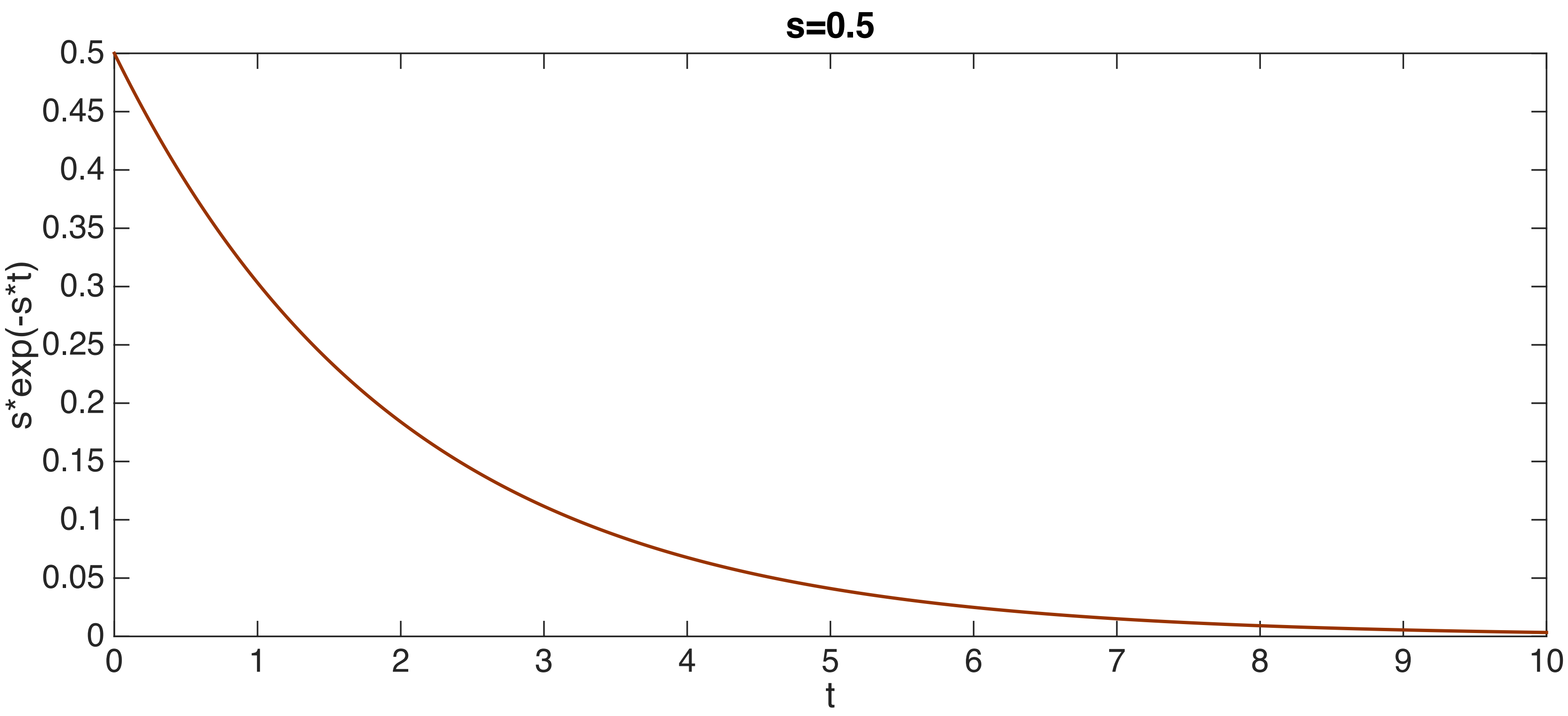
$$\int_{t=0^-}^{\infty} s e^{-st}\, \mathrm{d}t = 1 \qquad \text{for all } s > 0$$

- Taking $s = 1/4$, $s = 1/2$, $s = 1$, $s = 10$, and $s = 50$ we find

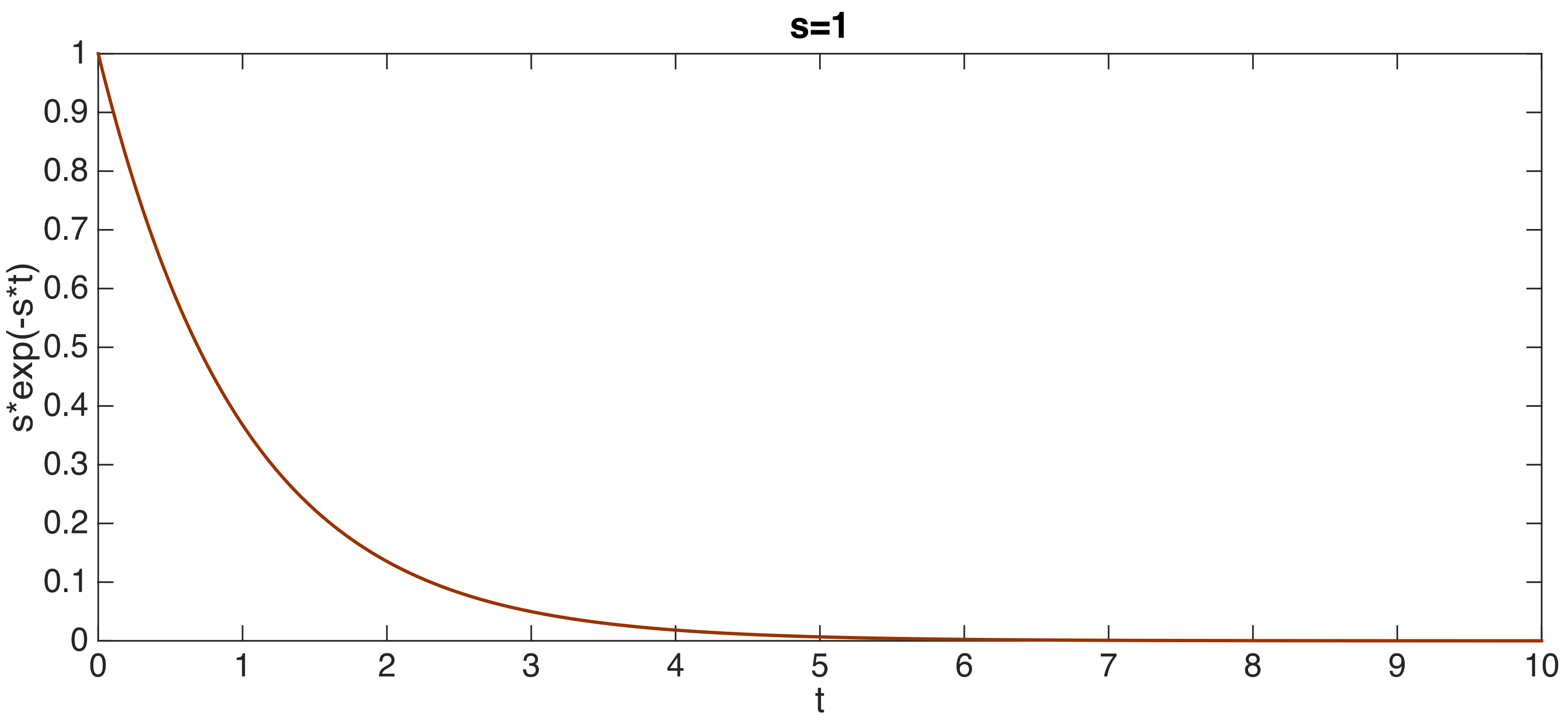
The Laplace Transform



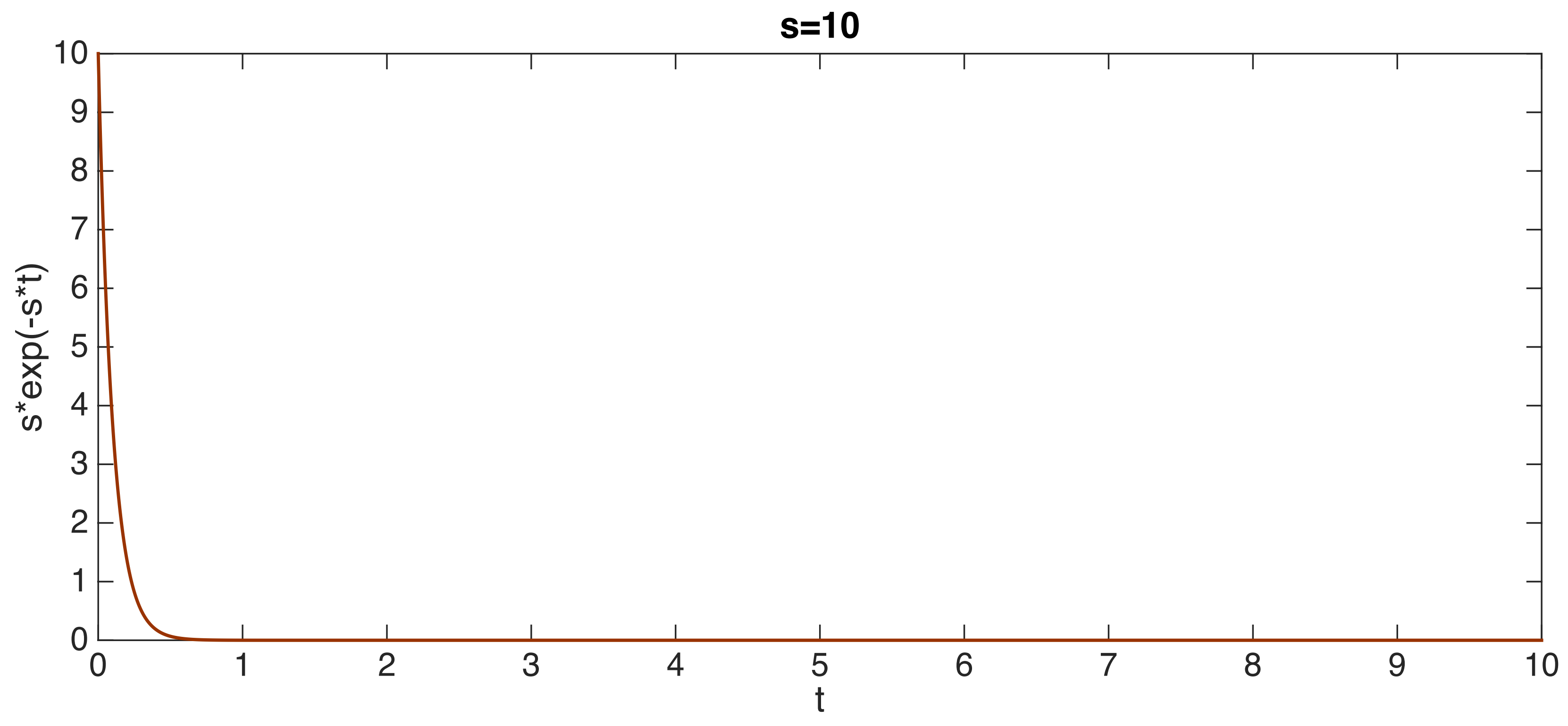
The Laplace Transform



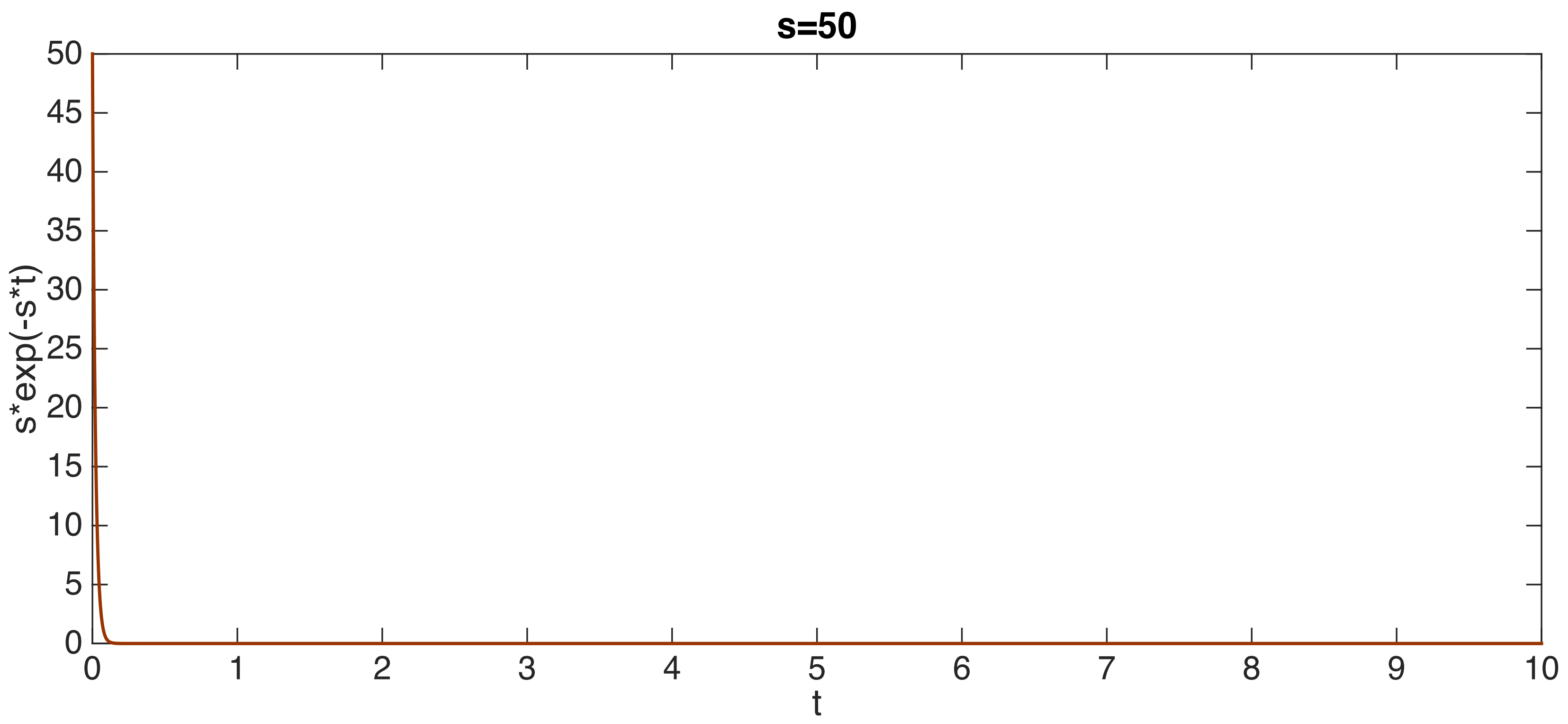
The Laplace Transform



The Laplace Transform



The Laplace Transform



- **Abel's final-value theorem**
- Let $X(s)$ denote the one-sided Laplace transform of $x(t)$, $s \in \mathrm{ROC}_x$
- Abel's final-value theorem states that

$$\lim_{s \to 0} sX(s) = \lim_{t \to \infty} x(t)$$

provided $\lim_{t \to \infty} x(t)$ exists

- Left-hand side: Laplace-domain
- Right-hand side: time-domain

The Laplace Transform

Niels Henrik Abel
Born 1802
Died 1829

The Laplace Transform

One-Sided Laplace Transforms

Time signal	**One-sided Laplace transform**	**ROC**	**parameters**
$e^{at}u(t)$	$\frac{1}{s-a}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$	$a \in \mathbb{C}$
$-e^{at}u(-t)$	0	$\mathbb{C}$	$a \in \mathbb{C}$
$\frac{t^{k-1}e^{at}}{(k-1)!}u(t)$	$\frac{1}{(s-a)^k}$	$\mathrm{Re}(s) > \mathrm{Re}(a)$	$a \in \mathbb{C},\ k \in \mathbb{N}$
$-\frac{t^{k-1}e^{at}}{(k-1)!}u(-t)$	0	$\mathbb{C}$	$a \in \mathbb{C},\ k \in \mathbb{N}$
$e^{at}\cos(\Omega_0 t)u(t)$	$\frac{s-a}{(s-a)^2+\Omega_0^2}$	$\mathrm{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$e^{at}\sin(\Omega_0 t)u(t)$	$\frac{\Omega_0}{(s-a)^2+\Omega_0^2}$	$\mathrm{Re}(s) > a$	$a, \Omega_0 \in \mathbb{R}$
$\delta(t)$	1	$\mathbb{C}$	–
$\delta'(t)$	s	$\mathbb{C}$	–

The Laplace Transform

Properties of the One-Sided Laplace Transform: $x_c(t) = x(t)u(t)$, $t \in \mathbb{R}$

Property	**Time signal**	**One-sided Laplace transform**	**ROC**	**Parameters**
Convolution	$y_c(t) = h_c(t) * x_c(t)$	$Y(s) = H(s)X(s)$	$\mathrm{ROC}_{h_c} \cap \mathrm{ROC}_{x_c}$	–
Diff. s-domain	$-tx(t)$	$\frac{\mathrm{d}X(s)}{\mathrm{d}s}$	ROC_{x_c}	–
Diff. t-domain	$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$	$sX(s) - x(0^-)$	ROC_{x_c}	–
Int. t-domain	$\int_{\tau=0^-}^{t} x(\tau)\,\mathrm{d}\tau$	$\frac{1}{s}X(s)$	$\{\mathrm{ROC}_{x_c} \vert \mathrm{Re}(s) > 0\}$	–
Shift t-domain	$x_c(t-\tau)$	$e^{-s\tau}X(s)$	ROC_{x_c}	$\tau \in \mathbb{R}$, $\tau > 0$
Shift s-domain	$e^{at}x(t)$	$X(s-a)$	$s - a \in \mathrm{ROC}_{x_c}$	$a \in \mathbb{C}$
Scaling	$x(at)$	$\frac{1}{a}X\left(\frac{s}{a}\right)$	$s/a \in \mathrm{ROC}_{x_c}$	$a \in \mathbb{R}$, $a > 0$

- **Circuit Theory Revisited**
- **KCL** Kirchhoff's current law: the algebraic sum of all branch currents flowing into any node must be zero
- For a node with N branches

$$\sum_{n=1}^{N} i_n(t) = 0$$

- **KVL** Kirchhoff's voltage law: the algebraic sum of the branch voltages around any closed path in a network must be zero
- For a closed path consisting of N branches

$$\sum_{n=1}^{N} v_n(t) = 0$$

The Laplace Transform

- Let

$I_n(s)$ be the one-sided Laplace transform of $i_n(t)$

$n = 1, 2, \ldots, N$

$V_n(s)$ be the one-sided Laplace transform of $V_n(t)$

$n = 1, 2, \ldots, N$

- Since the Laplace transform is linear, we have

- **KCL** Kirchhoff's current law in the Laplace domain:

$$\sum_{n=1}^{N} I_n(s) = 0$$

- **KVL** Kirchhoff's voltage law in the Laplace domain:

$$\sum_{n=1}^{N} V_n(s) = 0$$

- **Constitutive relations**
- *Resistor*

$$v(t) = R\,i(t) \quad \text{with one-sided Laplace transform} \quad V(s) = R\,I(s)$$

- *Capacitor*

$i(t) = C\frac{\mathrm{d}v(t)}{\mathrm{d}t}$ with one-sided Laplace transform $I(s) = sC\,V(s) - C\,v(0^-)$

- *Inductor*

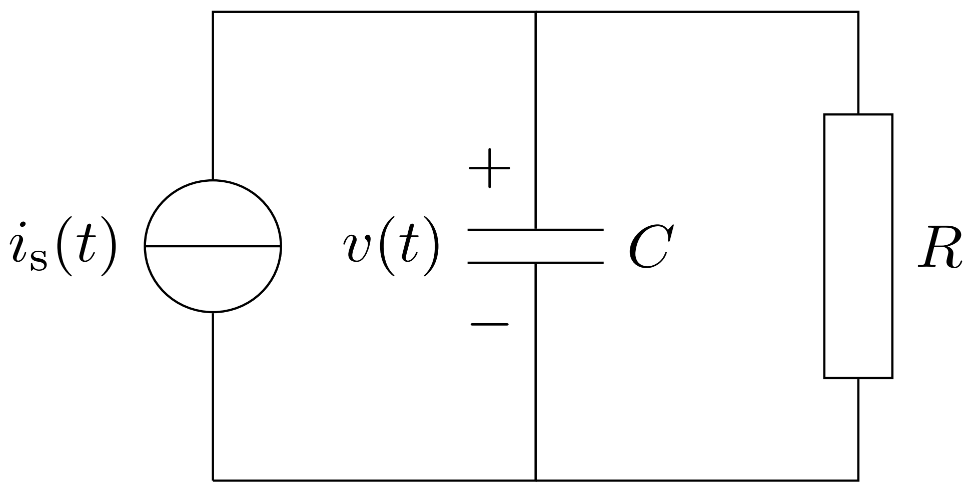
$$v(t) = L\frac{\mathrm{d}i(t)}{\mathrm{d}t} \quad \text{with one-sided Laplace transform} \quad V(s) = sL\,I(s) - Li(0^-)$$

- For circuits with vanishing initial conditions (the circuit is initially at rest), we define the Laplace impedance $Z(s)$ through the relation

$$V(s) = Z(s)\,I(s)$$

Resistor: $Z(s) = R$, **Capacitor**: $Z(s) = \dfrac{1}{sC}$, **Inductor**: $Z(s) = sL$

- **Example.** Consider the circuit sketched below
- Input signal: $i_s(t) = I_0\delta(t)$
- Output signal: $v(t)$
- The circuit is initially at rest



- Kirchhoff's current law in the time-domain:

$$C\frac{\mathrm{d}v}{\mathrm{d}t} + R^{-1}v(t) = I_0\delta(t), \qquad t > 0^-$$

$v(0^-) = 0$

The Laplace Transform

- Kirchhoff's current law in the s-domain:

$$sCV(s) + R^{-1}V(s) = I_0$$

- Divide by C to obtain

$$\left(s + \frac{1}{\tau}\right)V(s) = \frac{I_0}{C}, \qquad \tau = RC$$

- We find

$$V(s) = \frac{I_0}{C}\frac{1}{s + \frac{1}{\tau}}, \qquad \tau = RC$$

The Laplace Transform

- Using the table for one-sided Laplace transforms, the voltage is found as

$$v(t) = \frac{I_0}{C} e^{-t/\tau} u(t), \qquad \tau = RC$$

- The current through the capacitor follows as

$$i_c(t) = C \frac{\mathrm{d}v(t)}{\mathrm{d}t} = I_0 \left[\delta(t) - \frac{1}{\tau} e^{-t/\tau} u(t) \right], \qquad \tau = RC$$

- Observe that we can also write

$$I_c(s) = \frac{Y_{\text{cap}}(s)}{Y_{\text{cap}}(s) + Y_{\text{res}}(s)} I_0$$

- $Y_{\text{cap}}(s) = sC$ and $Y_{\text{res}}(s) = R^{-1}$ are the Laplace domain *admittances* of the capacitor and resistor, respectively $(Y(s) = Z^{-1}(s))$

- Substitution gives

$$I_c(s) = \frac{sC}{sC + R^{-1}} I_0 = \left(1 - \frac{1}{\tau}\frac{1}{s + \frac{1}{\tau}}\right) I_0, \qquad \tau = RC$$

- Using the table for the one-sided Laplace transform, we again arrive at

$$i_{\mathrm{c}}(t) = I_0 \left[\delta(t) - \frac{1}{\tau} e^{-t/\tau} u(t)\right], \qquad \tau = RC$$