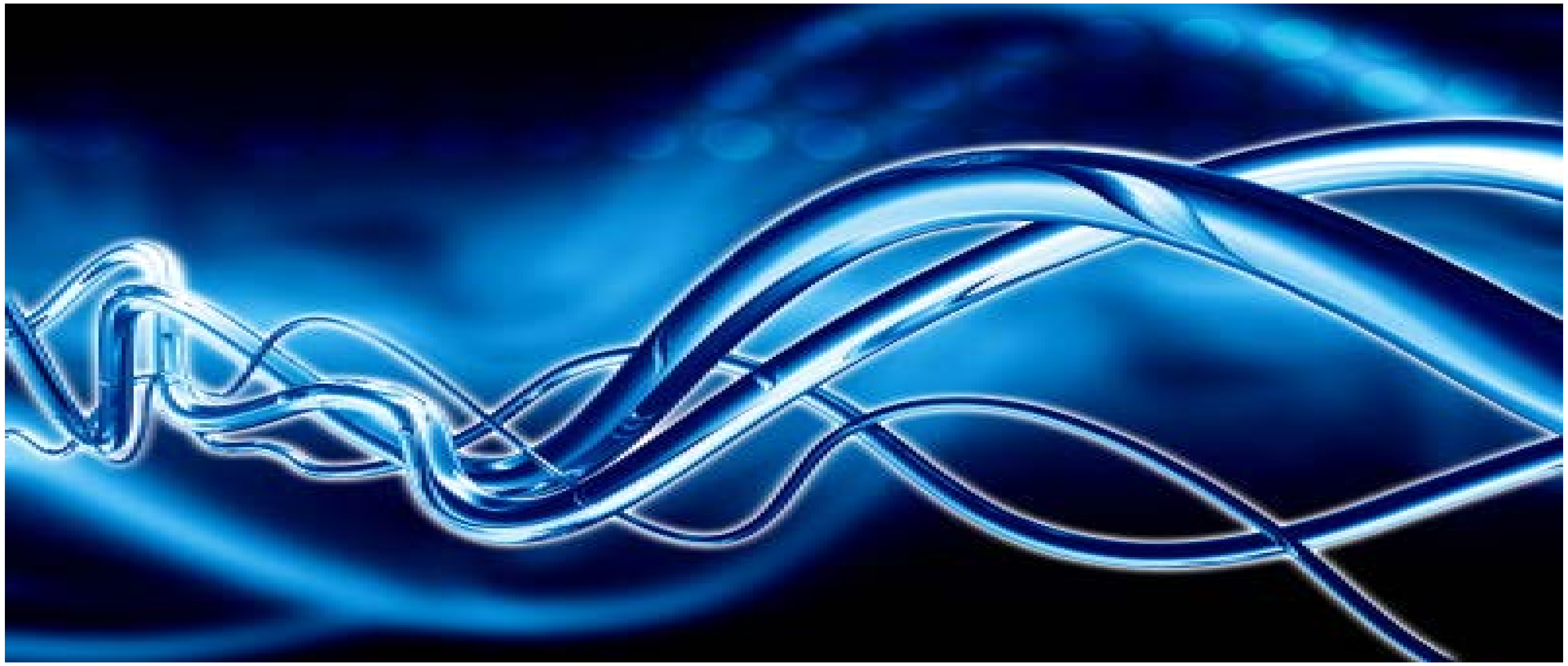
Signals and Systems



2. Linear and Time-Invariant Systems

Linear and Time-Invariant Systems

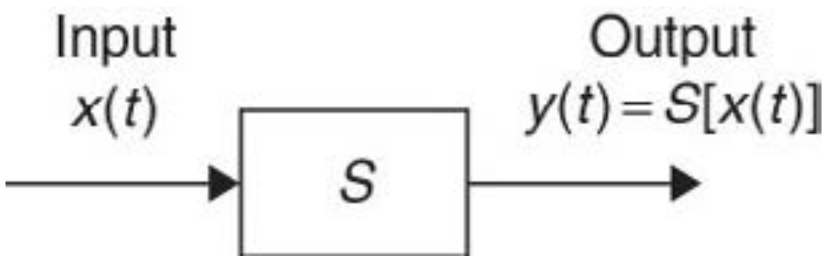
- **Contents**
  Linear and time-invariant systems
  The impulse response and the convolution integral
  Linear ordinary differential equations and LTI systems
  Causality
  BIBO stability

- **Book:**
  Chapter 2

- **Exercises:** 2.1, 2.2, 2.3, 2.4, 2.5, 2.8, 2.9, 2.11 (3rd Ed.)
  2.1, 2.2, 2.5, 2.7, 2.8, 2.12, 2.14, 2.18 (2nd Ed.)

Linear and Time-Invariant Systems

- We consider a system with a single input and a single output
- Input signal: $x(t)$, output signal $y(t)$
- Such systems are called SISO systems
- SISO stands for **S**ingle **I**nput **S**ingle **O**utput
- If a system has **M**ultiple **I**nputs and **M**ultiple **O**utputs it is called a MIMO system
- We restrict ourselves to SISO systems



Linear and Time-Invariant Systems

- The action of the system on the input signal $x(t)$ is described by the system operator S
- We write
$$y(t) = S\{x(t)\}$$
- In this course we are particularly interested in systems that are **L**inear and **T**ime-**I**nvariant
- Such systems are called LTI systems

Linear and Time-Invariant Systems

- **Linearity** Suppose we have two input signals $x_1(t)$ and $x_2(t)$. Denote the corresponding output signals by $y_1(t)$ and $y_2(t)$:

$$y_1(t) = S\{x_1(t)\} \quad \text{and} \quad y_2(t) = S\{x_2(t)\}$$

The system is called *linear* if

$$\begin{aligned} y(t) &= S\{\alpha x_1(t) + \beta x_2(t)\} \\ &= \alpha S\{x_1(t)\} + \beta S\{x_2(t)\} \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

for any two constants α and β

Linear and Time-Invariant Systems

- Any linear combination of input signals produces the same linear combination of their corresponding output signals
- Taking $\beta = 0$, it follows from the above definition that

$$y(t) = S\{\alpha x_1(t)\} = \alpha S\{x_1(t)\} = \alpha y_1(t)$$

- In other words, if you scale the input signal by a factor α, the output signal will scale with the same factor

Linear and Time-Invariant Systems

- **Example** Consider a SISO system with input signal $x(t)$ and an output signal given by
$$y(t) = \frac{1}{T}\int_{\tau=t-T}^{t} x(\tau)\,\mathrm{d}\tau + B,$$
where B is a constant. Such a system is called a biased averager (can you see why?)

- Scaling the input signal by a factor α, we obtain the output signal
$$\frac{\alpha}{T}\int_{\tau=t-T}^{t} x(\tau)\,\mathrm{d}\tau + B,$$
which is not equal to $\alpha y(t)$ unless $B = 0$

Linear and Time-Invariant Systems

- The averager is nonlinear for $B \neq 0$
- For $B = 0$, it is easy to see that a linear combination of input signals produces the same linear combination of the corresponding output signals
- The averager is linear for $B = 0$

Linear and Time-Invariant Systems

- **Time-Invariance** Let $y(t)$ be the output signal that corresponds to an input signal $x(t)$:

$$y(t) = S\{x(t)\}$$

- The system is called *time-invariant* if

$$y(t \pm \tau) = S\{x(t \pm \tau)\}$$

for any time shift $\tau > 0$

- In words: shifting your input signal produces an equally time-shifted output signal

Linear and Time-Invariant Systems

- Let the Dirac distribution be the input signal of an LTI system
- The corresponding output signal is written as $h(t)$ and is called the *impulse response*:

$$h(t) = S\{\delta(t)\}$$

- We claim that if you know the impulse response of an LTI system then you know the response to any other input signal!

Linear and Time-Invariant Systems

- To show this, let $y(t)$ be the output signal that corresponds to an input signal $x(t)$:

$$y(t) = S\{x(t)\}$$

- Because of the sifting property of the Dirac distribution, we have

$$x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau$$

- The right-hand side of the above expression can be seen as a continuous weighted summation of shifted Dirac distributions

Linear and Time-Invariant Systems

- Substitution gives

$$y(t) = S\left\{\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)\,\mathrm{d}\tau\right\}$$

- Now note that S is linear and acts on functions that depend on time t
- This allows us to write

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)S\{\delta(t-\tau)\}\,\mathrm{d}\tau$$

Linear and Time-Invariant Systems

- Since the system is time-invariant as well, we have

$$h(t-\tau) = S\{\delta(t-\tau)\}$$

- and we arrive at

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau$$

- Knowing the impulse response $h(t)$, we can determine the response $y(t)$ to any input signal $x(t)$ by evaluating the above integral

Linear and Time-Invariant Systems

- This integral is called a *convolution integral*
- Short-hand notation:

$$y = x * h \quad \text{or} \quad y(t) = x(t) * h(t)$$

- The asterisk is called the *convolution product*
- The output signal $y(t)$ is equal to the convolution product of the input signal $x(t)$ and the impulse response $h(t)$

Linear and Time-Invariant Systems

- For two real numbers a and b, we have

$$ab = ba$$

- The product of two real numbers commutes
- Is this also true for the convolution product? In other words, do we have

$$x * h = h * x?$$

- The answer is yes. Let's check it.

$$
\begin{aligned}
y(t) &= x * h \\
&= \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau \overset{p=t-\tau}{=} \int_{p=-\infty}^{\infty} x(t-p)h(p)\,\mathrm{d}p \\
&= \int_{p=-\infty}^{\infty} h(p)x(t-p)\,\mathrm{d}p = h * x
\end{aligned}
$$

- Conclusion: the convolution product of two signals commutes (due to the minus sign in the argument of h)

Linear and Time-Invariant Systems

- If you change the minus sign into a plus sign you get what is called the *cross-correlation* of the two signals $x(t)$ and $h(t)$ provided these signals are both real-valued:

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t+\tau)\,\mathrm{d}\tau = x \star h$$

- The cross correlation of two signals does *not* commute

$$x \star h \neq h \star x$$

- For the product of real numbers, there exists an identity element called “one" and written as 1 for which

$$a = a \cdot 1 = 1 \cdot a$$

- What is the identity element for the convolution product?
- We already know the answer to this question
- It is the Dirac distribution!

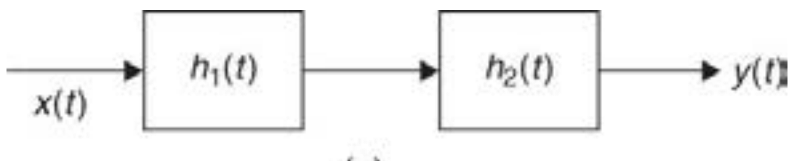
$$x = x * \delta = \delta * x$$

- The convolution product is also *associative*, that is, for three signals u, v, and w, we have (check this yourself)

$$(u * v) * w = u * (v * w)$$

Linear and Time-Invariant Systems

- This property can be exploited to determine the total impulse function of two LTI systems interconnected in cascade
- System 1: input signal $x(t)$, impulse function $h_1(t)$, output signal $u(t)$
- System 2: input signal $u(t)$, impulse function $h_2(t)$, output signal $y(t)$
- We assume that System 2 does not “load" System 1



- Response of the total system:

$$y = u * h_2 = (x * h_1) * h_2 = x * (h_1 * h_2) = x * h$$

- where we have introduced the impulse function of the total system as

$$h = h_1 * h_2 = h_2 * h_1$$

- Note that since the convolution product of two signals commute, we can interchange the order of the subsystems without affecting the output signal $y(t)$ (provided both systems do not “load" each other)

- **Computing the convolution integral: Example 1**
- We compute the convolution of the rectangular pulse p with itself
- Recall that

$$p(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- By definition, we have

$$y(t) = p * p = \int_{\tau=-\infty}^{\infty} p(\tau)\, p(t-\tau)\, \mathrm{d}\tau$$

- Since p vanishes outside the interval $(0,1)$, the integral simplifies to

$$y(t) = \int_{\tau=0}^{1} p(t-\tau)\,\mathrm{d}\tau$$

- It is convenient to rewrite this integral in such a way that the time coordinate t appears in the integration limits of the integral
- We use the substitution $t' = t - \tau$ to achieve this and arrive at

$$y(t) = \int_{t'=t-1}^{t} p(t')\,\mathrm{d}t'$$

- Now observe that for $t < 0$ we integrate over an interval outside the support of p. Consequently,

$$y(t) = 0 \quad \text{for } t < 0$$

- Similarly, for $t - 1 > 1$ we again integrate over an interval outside the support of p. We have

$$y(t) = 0 \quad \text{for } t > 2$$

Linear and Time-Invariant Systems

- For $0 < t < 1$ the lower bound falls outside of the support of p, while the upper bound belongs to this support. We have

$$y(t) = \underbrace{\int_{t'=t-1}^{0} p(t')\,\mathrm{d}t'}_{=0} + \int_{t'=0}^{t} p(t')\,\mathrm{d}t' = \int_{t'=0}^{t} \mathrm{d}t' = t \quad \text{for } 0 < t < 1$$

- Finally, for $1 < t < 2$ the upper bound falls outside of the support of p and the lower bound is in the support of p. In this case, we have

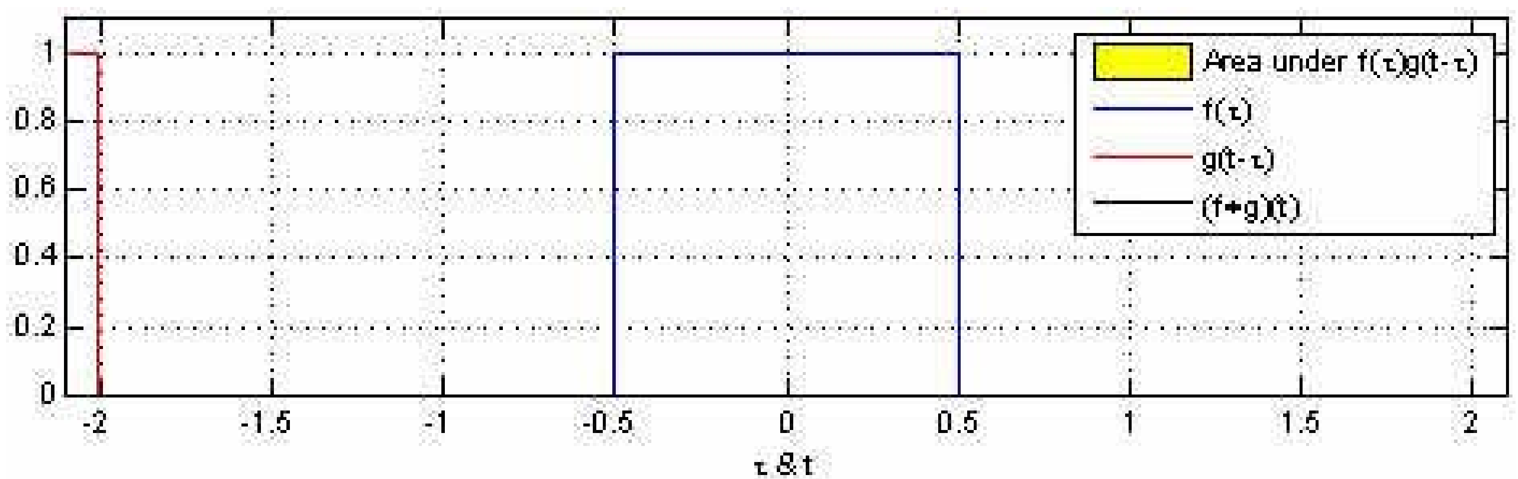
$$y(t) = \int_{t'=t-1}^{1} p(t')\,\mathrm{d}t' + \underbrace{\int_{t'=1}^{t} p(t')\mathrm{d}t'}_{=0} = \int_{t'=t-1}^{1} \mathrm{d}t' = 2 - t \quad \text{for } 1 < t < 2$$

Linear and Time-Invariant Systems

- Putting everything together, we find

$$y(t) = \Lambda(t)$$

- The convolution of the rectangular pulse with itself produces the triangular pulse function



Animation by Brian Amberg and adapted by Tinos (Wikipedia)

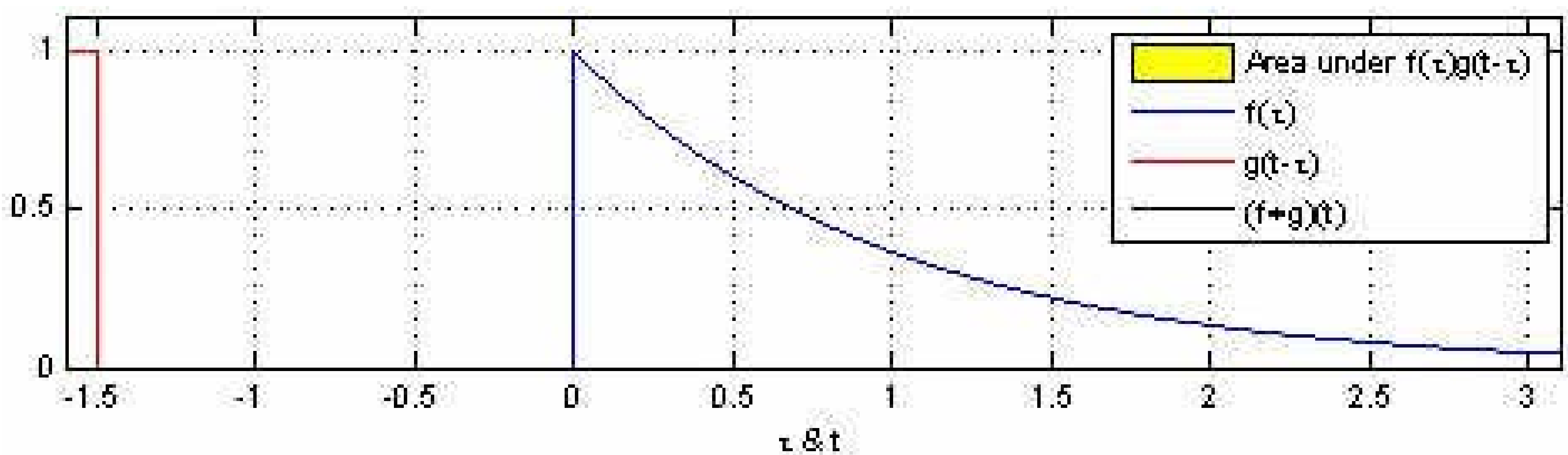
- **Computing the convolution integral: Example 2**
- We graphically determine the convolution of $p(t)$ and $p(t/10)$ (we use the blackboard for this)
- Finally, if the support of a signal x is (ℓ_x, u_x) and the support of a signal h is (ℓ_h, u_h) then

$$\text{the support of } y(t) = x(t) * h(t) \text{ is } (\ell_x + \ell_h, u_x + u_h)$$

Verify this statement!

Linear and Time-Invariant Systems

Another example:



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Linear and Time-Invariant Systems

- Up till now we have been looking at fairly general systems whose action on the input signal is described by some operator S

- Let us now be more specific and consider systems described by the linear ordinary differential equation

$$\left(a_N \frac{\mathrm{d}^N}{\mathrm{d}t^N} + a_{N-1} \frac{\mathrm{d}^{N-1}}{\mathrm{d}t^{N-1}} + \ldots + a_1 \frac{\mathrm{d}}{\mathrm{d}t} + a_0\right) y(t) = \left(b_M \frac{\mathrm{d}^M}{\mathrm{d}t^M} + b_{M-1} \frac{\mathrm{d}^{M-1}}{\mathrm{d}t^{M-1}} + \ldots + b_1 \frac{\mathrm{d}}{\mathrm{d}t} + b_0\right) x(t)$$

  which holds for $t > 0$

- N and M are positive integers

- $x(t)$ is the prescribed input signal
- $y(t)$ is the desired output signal
- To obtain the output signal $y(t)$, we also need the N initial conditions

$$y(0) \quad \text{and} \quad \left.\frac{\mathrm{d}^k y(t)}{\mathrm{d}t^k}\right|_{t=0} \quad \text{for } k = 1, 2, \ldots, N-1$$

- RLC circuits, mechanical systems, etc. can all be described by a differential equation of the above form

Linear and Time-Invariant Systems

- Further on we will show you how to solve the differential equation using the Laplace transform
- For now it suffices to say that the solution $y(t)$ is given by

$$y(t) = y_{\text{zs}}(t) + y_{\text{zi}}(t)$$

- $y_{\text{zs}}(t)$ is called the *zero-state* response. This is the solution exclusively due to the input with the initial conditions set to zero
- $y_{\text{zi}}(t)$ is called the *zero-input* response. This is the solution exclusively due to the initial conditions with the input set to zero

Linear and Time-Invariant Systems

- For vanishing initial conditions the system is linear and time-invariant (LTI). This can easily be seen from the differential equation (check this for yourself)

- In this case the zero-input response vanishes and the solution is equal to the zero-state response

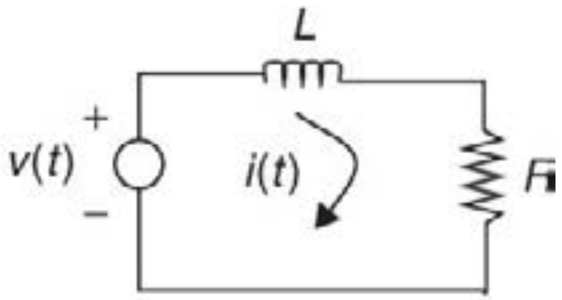
- For nonvanishing initial conditions, the system is no longer an LTI system

- **Example.** Consider a circuit consisting of a resistor of R in series with an inductor L and a voltage source $v(t) = Bu(t)$. The initial current in the inductor is I_0. The input signal of the system is $v(t)$, the current $i(t)$ in the circuit is the output signal.

- From Kirchhoff's voltage law:

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + Ri(t) = v(t) \quad t > 0$$

with initial condition $i(0) = I_0$



Linear and Time-Invariant Systems

- The output signal is given by

$$i(t) = i_{\text{zs}}(t) + i_{\text{zi}}(t), \qquad t > 0$$

- with

$$i_{\text{zs}}(t) = \frac{B}{R}(1 - e^{-t/\tau}), \quad i_{\text{zi}}(t) = I_0 e^{-t/\tau}, \quad \text{and} \quad \tau = L/R$$

- If we double the amplitude of the input signal then the output signal becomes

$$i(t) = 2i_{\text{zs}}(t) + i_{\text{zi}}(t)$$

with $i_{\text{zs}}(t)$ and $i_{\text{zi}}(t)$ as above

Linear and Time-Invariant Systems

- We observe that the output is *not* doubled, since $i_{\text{zi}}(t)$ does not vanish
- However, for $I_0 = 0$ (vanishing initial condition) we do have $i_{\text{zi}}(t) = 0$ and the output *is* doubled in this case
- With vanishing initial conditions, the system is linear

- A continuous-time system S is *causal* if
    * whenever its input $x(t) = 0$, and there are no initial conditions, the output is $y(t) = 0$
    * the output $y(t)$ does not depend on future inputs

Linear and Time-Invariant Systems

- An LTI system is causal if

$$h(t) = 0 \quad \text{for } t < 0 \qquad \text{(causal LTI system)}$$

- Indeed, for an LTI system we have the convolution integral

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

- and writing this integral as

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)h(t-\tau)\,\mathrm{d}\tau + \int_{t}^{\infty} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

Linear and Time-Invariant Systems

- we observe that in the second integral integration takes place over future inputs
- For a causal LTI system, these inputs cannot contribute to the output signal at time instant t
- Consequently, for a causal system we must have $h(t-\tau)=0$ for $t<\tau<\infty$ or $h(t)=0$ for $t<0$
- In case the LTI system is causal we are left with

$$y(t)=\int_{\tau=-\infty}^{t} x(\tau)\,h(t-\tau)\,\mathrm{d}\tau$$

- In addition, if the input signal also vanishes prior to $t = 0$, that is, if $x(t) = 0$ for $t < 0$, then the convolution integral simplifies even further. In this case we have

$$y(t) = \int_{\tau=0}^{t} x(\tau)h(t-\tau)\,\mathrm{d}\tau$$

- Finally, we discuss the concept of BIBO stability
- BIBO stands for **B**ounded **I**nput **B**ounded **O**utput
- We are given a bounded input signal $x(t)$, that is, a signal that satisfies

$$|x(t)| \leq M$$

  for some positive M

- We ask: Under what condition(s) is the output signal $y(t)$ also bounded?

- To answer this question, consider

$$\begin{aligned}|y(t)| &= \left|\int_{\tau=-\infty}^{\infty} x(t-\tau)\,h(\tau)\,\mathrm{d}\tau\right| \\ &\leq \int_{\tau=-\infty}^{\infty} |x(t-\tau)|\,|h(\tau)|\,\mathrm{d}\tau \\ &\leq M\int_{\tau=-\infty}^{\infty} |h(\tau)|\,\mathrm{d}\tau\end{aligned}$$

- From this inequality it follows that if

$$\int_{\tau=-\infty}^{\infty} |h(\tau)|\,\mathrm{d}\tau < \infty$$

  then the output signal $y(t)$ is bounded as well

- If the impulse response is absolutely integrable (the action of the impulse response is finite) then the output is bounded as well
- An LTI system is called *BIBO stable* if the impulse response is absolutely integrable