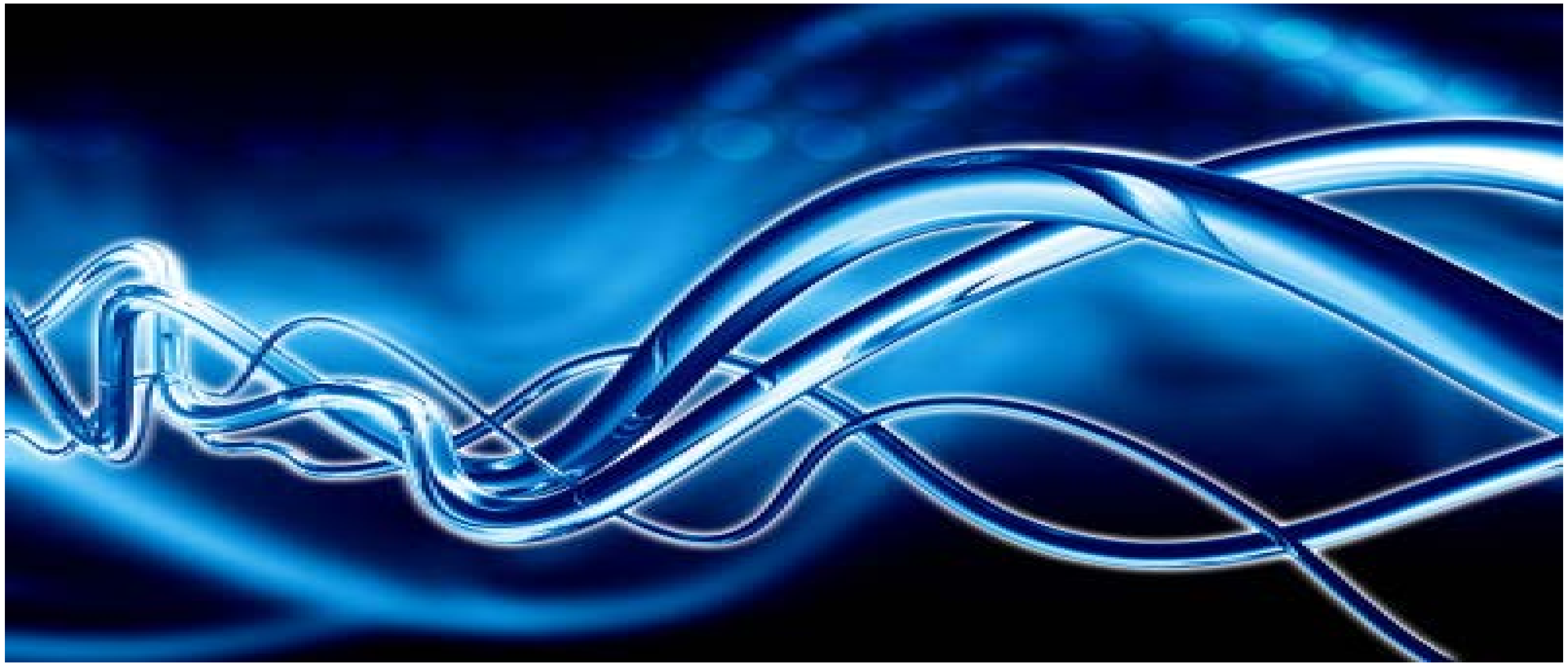
Signals and Systems



1. Standard Signals

Standard Signals

- **Contents**
  Standard signals
  Even and odd signals
  The energy and power of continuous-time signals
  Periodic signals
  The Dirac distribution

- **Book:**
  Chapter 1

- **Exercises:** 1.1, 1.2, 1.3, 1.4, 1.6, 1.9, 1.12 (3rd Ed.)
  1.1, 1.3, 1.4, 1.6, 1.8, 1.13, 1.16 (2nd Ed.)

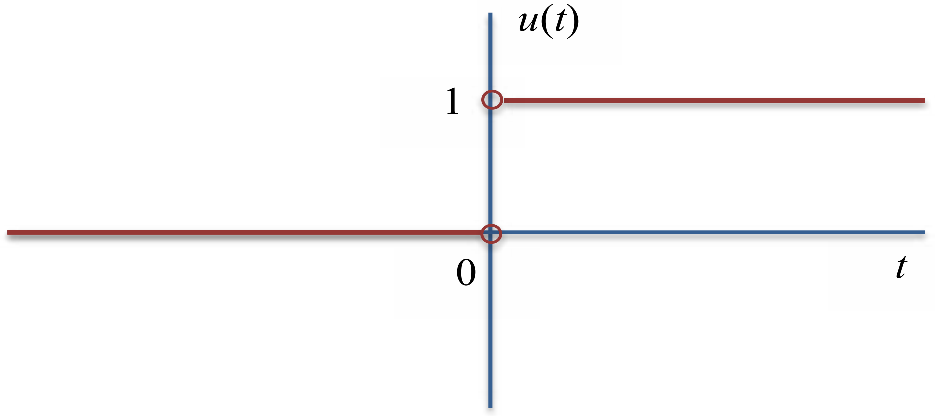
  Additional exercises at the end of this lecture

Standard signals

- The *Heaviside unit step function*

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

- The step function can be used to model switch-on phenomena
- The step function $u(-t)$ can be used to model switch-off phenomena



Standard Signals

Oliver Heaviside
Born 1850
Died 1925

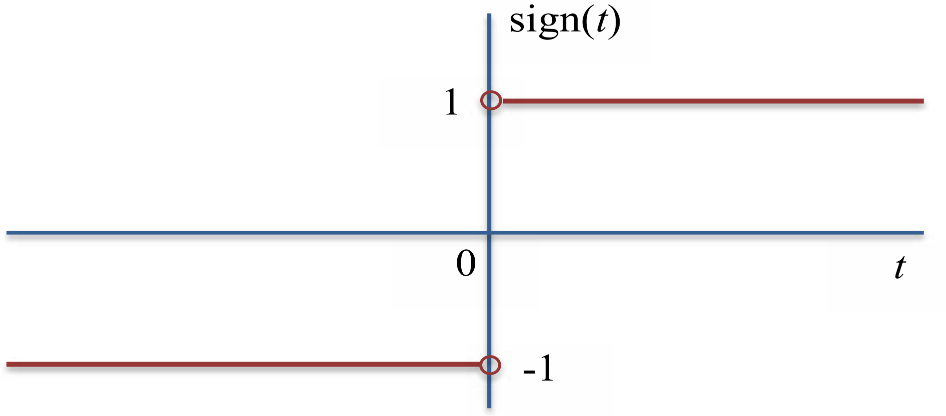
Standard signals

- The *sign* or *signum function*

$$\text{sign}(t) = \begin{cases} -1 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

- The sign function in terms of unit step functions

$$\text{sign}(t) = 2u(t) - 1 \quad \text{or} \quad \text{sign}(t) = u(t) - u(-t)$$



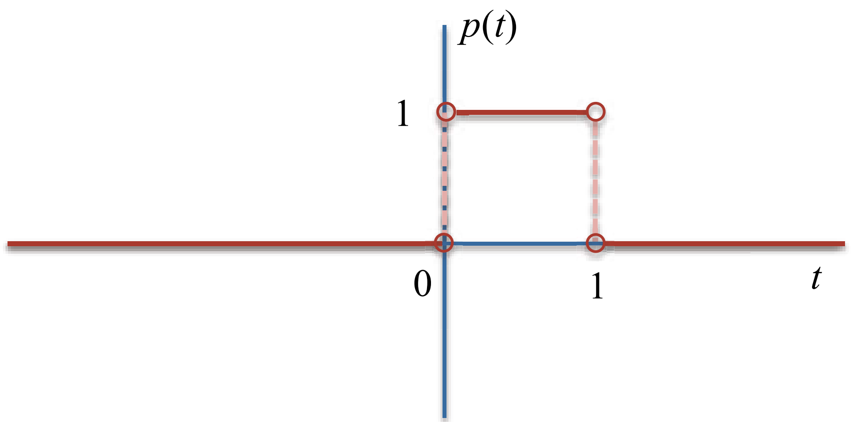
Standard signals

- The *rectangular pulse function*

$$p(t) = \begin{cases} 1 & \text{for } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

- The pulse function in terms of unit step functions

$$p(t) = u(t) - u(t-1)$$



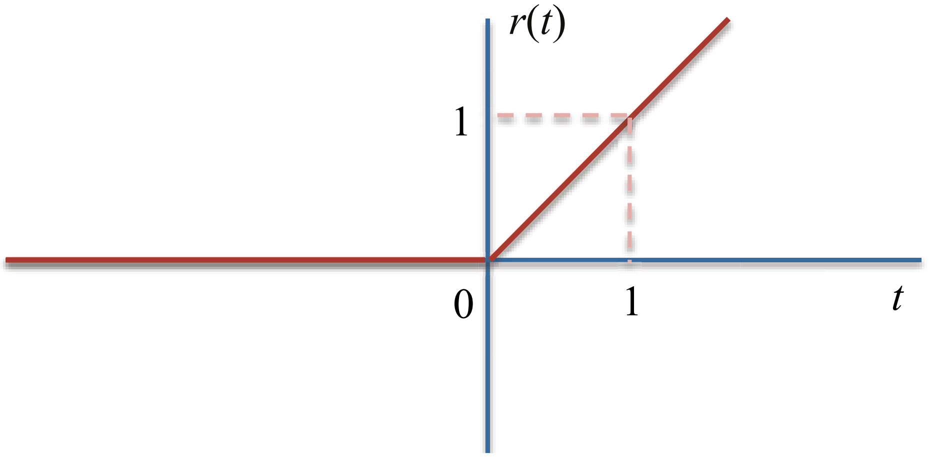
Standard signals

- The *ramp function*

$$r(t) = \begin{cases} t & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

- The ramp function in terms of the unit step function

$$r(t) = t\,u(t)$$



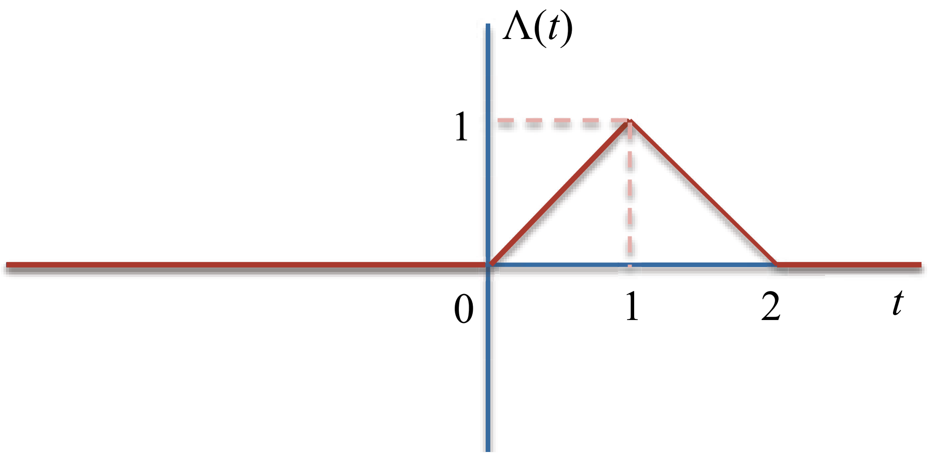
Standard signals

- The *triangular pulse function*

$$\Lambda(t) = \begin{cases} t & \text{for } 0 \le t \le 1 \\ 2 - t & \text{for } 1 < t \le 2 \\ 0 & \text{otherwise} \end{cases}$$

- The triangular pulse function in terms of ramp functions

$$\Lambda(t) = r(t) - 2r(t-1) + r(t-2)$$



Standard signals

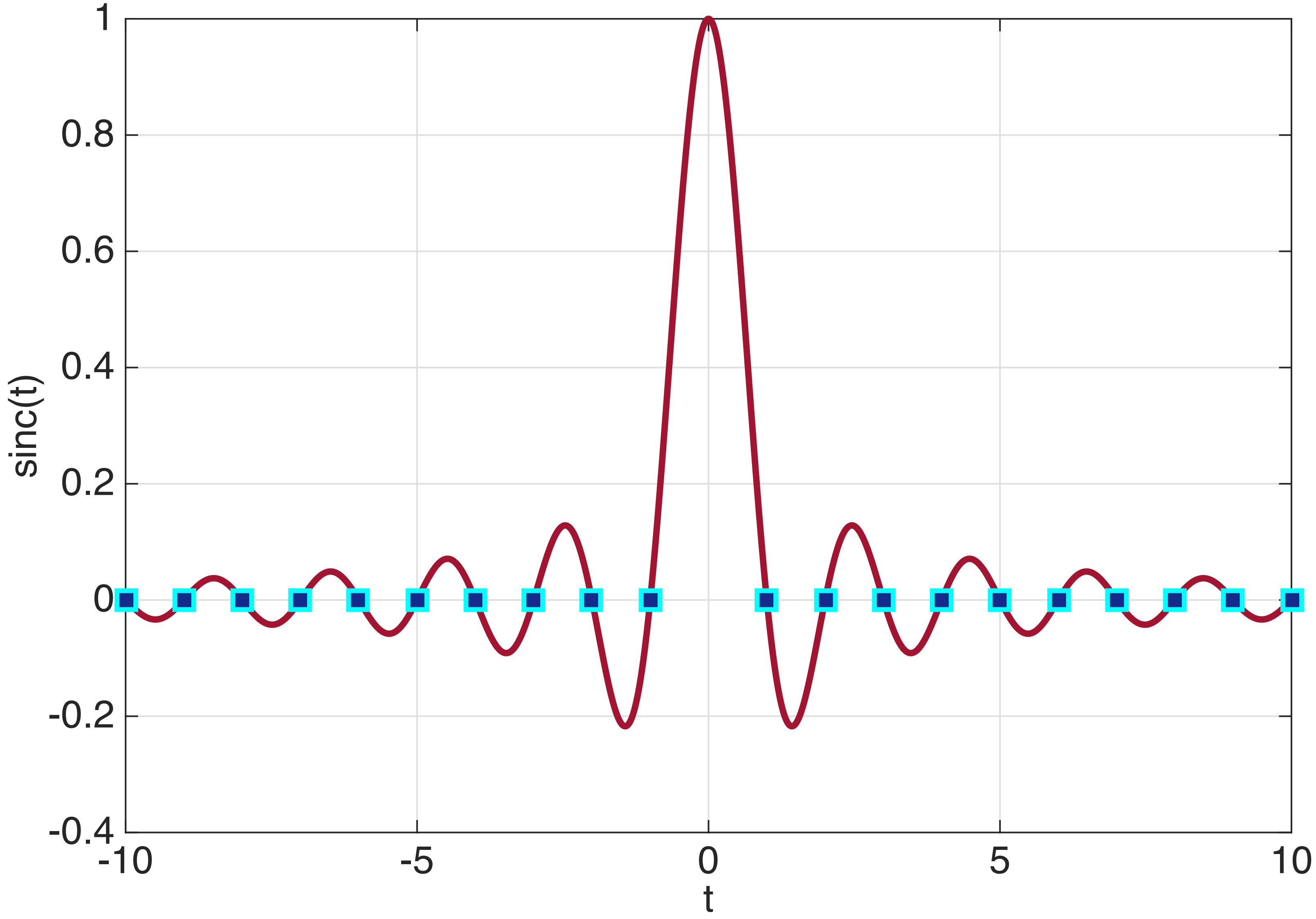
- The *sinc function*

$$S(t) = \operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \qquad t \in \mathbb{R}$$

- Some properties:
  - $S(0) = 1$
  - $S(n) = 0$, n a nonzero integer
  - Integral:

$$\int_{t=-\infty}^{\infty} S(t)\,\mathrm{d}t = 1$$

Standard signals



Even and odd signals

- A continuous-time signal $x(t)$ is called *even* if

$$x(-t) = x(t) \quad \text{for all } t \in \mathbb{R}$$

- A continuous-time signal $x(t)$ is called *odd* if

$$x(-t) = -x(t) \quad \text{for all } t \in \mathbb{R}$$

Even and odd signals

- A signal $y(t)$ defined on the entire t-axis can be written as a superposition of an even signal $y_{\mathrm{e}}(t)$ and an odd signal $y_{\mathrm{o}}(t)$:

$$y(t) = y_{\mathrm{e}}(t) + y_{\mathrm{o}}(t)$$

with

$$y_{\mathrm{e}}(t) = \frac{y(t) + y(-t)}{2} \quad \text{and} \quad y_{\mathrm{o}}(t) = \frac{y(t) - y(-t)}{2}$$

Energy of a continuous-time signal

- The *energy* of a continuous-time signal $x(t)$ is defined as

$$E_x = \int_{t=-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t$$

- A continuous-time signal $x(t)$ is called a *finite-energy signal* or *square integrable* if its energy is finite, that is, if

$$E_x < \infty \quad \text{or} \quad \int_{t=-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t < \infty$$

The action of a continuous-time signal

- The integral

$$\int_{t=-\infty}^{\infty} |x(t)|\,\mathrm{d}t$$

is sometimes called the *action* of the continuous-time signal $x(t)$. If the action is finite, that is, if

$$\int_{t=-\infty}^{\infty} |x(t)|\,\mathrm{d}t < \infty$$

the signal is called *absolutely integrable*

The power of a continuous-time signal

- The *power* of a continuous-time signal $x(t)$ is defined as

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{t=-T}^{T} |x(t)|^2 \, \mathrm{d}t$$

- From this definition it immediately follows that a finite-energy signal has zero power:

$$P_x = 0 \quad \text{for a finite-energy signal } x(t)$$

Periodic signals

- A continuous-time signal $x(t)$ is called *periodic* if there exists a $T > 0$ called a *period* of $x(t)$ such that

$$x(t+T) = x(t) \quad \text{for every } t \in \mathbb{R}$$

- A period of a periodic signal is not unique
- If T is a period, then $2T, 3T, \ldots$ are also periods of $x(t)$
- The smallest period T of $x(t)$ is called the *fundamental period* and is denoted as T_0
- The fundamental period T_0 is unique

Periodic signals

- Suppose we are given two periodic signals $x(t)$ and $y(t)$
- Signal $x(t)$ has a fundamental period T_0
- Signal $y(t)$ has a fundamental period T_1

Periodic signals

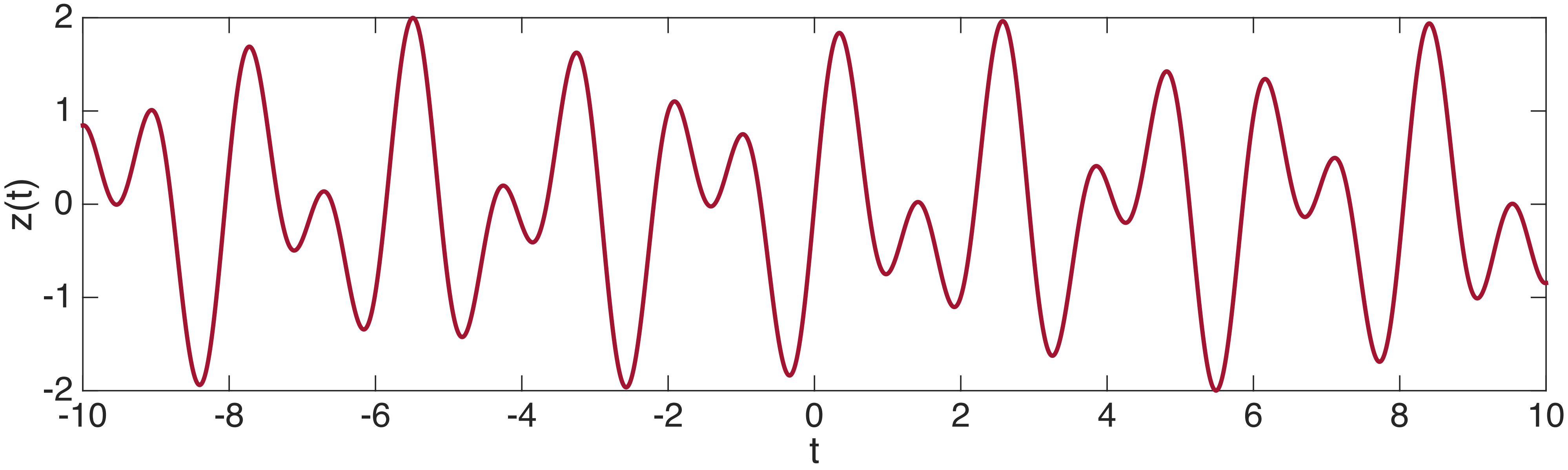
- Now consider the sum of these two periodic signals

$$z(t) = x(t) + y(t)$$

- $z(t)$ is periodic if $M \cdot T_1$ periods of $y(t)$ can be exactly included into $N \cdot T_0$ periods of $x(t)$
- The fundamental period of $z(t)$ is then the least common multiple of T_0 and T_1

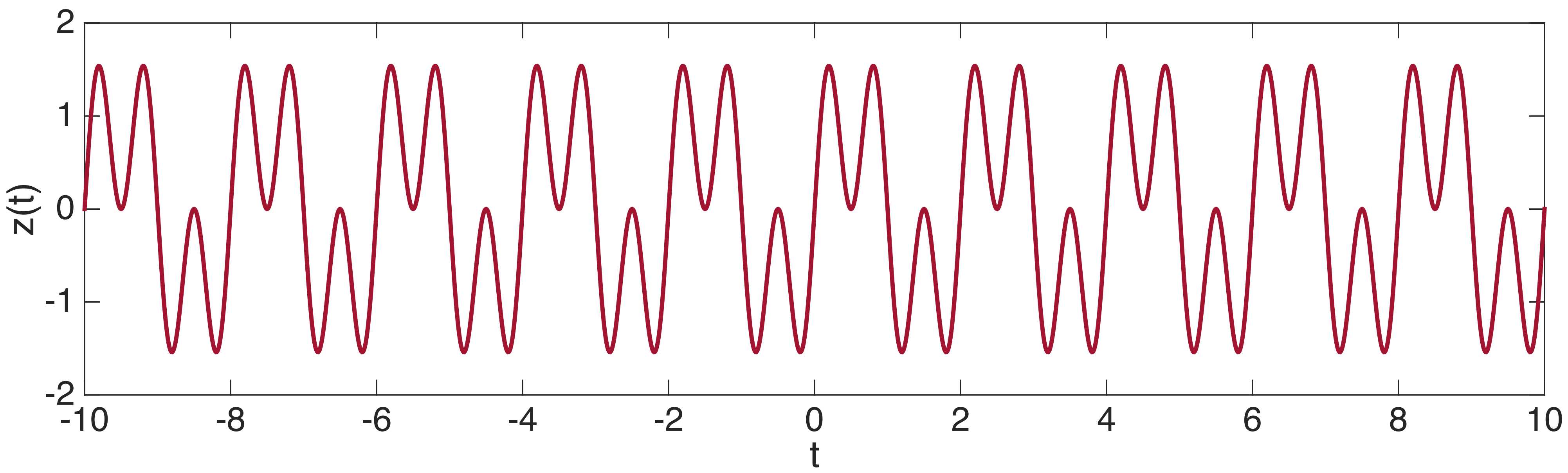
Periodic signals

- **Example 1:** $x(t) = \sin(\sqrt{3}\pi t)$ and $y(t) = \sin(\pi t)$. The signal $z(t) = x(t) + y(t)$ is not periodic



Periodic signals

- **Example 2:** $x(t) = \sin(\pi t)$ and $y(t) = \sin(3\pi t)$. In this case $T_0 = 2$ s and $T_1 = 2/3$ s and $z(t)$ is periodic. The fundamental period of $z(t)$ is $3T_1 = T_0$.



Energy and power of periodic signals

- Let $x(t)$ denote a continuous-time periodic signal with fundamental period T_0
- Since $x(t)$ is periodic, $|x(t)|^2$ is periodic as well and consequently E_x is infinite

A periodic signal is an infinite energy signal

- What about the power?

Energy and power of periodic signals

- To answer this question, we first have another look at the energy integral
- Since we know that the signal is periodic with fundamental period T_0, we compute the energy integral as follows
- First, consider the integral

$$E_x^{(N)} = \int_{t=t_0-NT_0}^{t_0+NT_0} |x(t)|^2 \, \mathrm{d}t,$$

- where t_0 is an arbitrary fixed time instant and N a positive integer

Energy and power of periodic signals

- The length of the integration interval is $2NT_0$ and the energy and power of the periodic signal follow as

$$E_x = \lim_{N\to\infty} E_x^{(N)} \quad \text{and} \quad P_x = \lim_{N\to\infty} \frac{1}{2NT_0} E_x^{(N)}$$

- Using the periodicity of $x(t)$, we find that

$$E_x^{(N)} = 2N \int_{t=t_0}^{t_0+T_0} |x(t)|^2 \, \mathrm{d}t$$

- Clearly, $E_x^{(N)}$ grows linearly in N as N increases and the limit $\lim_{N\to\infty} E_x^{(N)}$ does not exists (as claimed above)

Energy and power of periodic signals

- The power, however, does exist and is given by

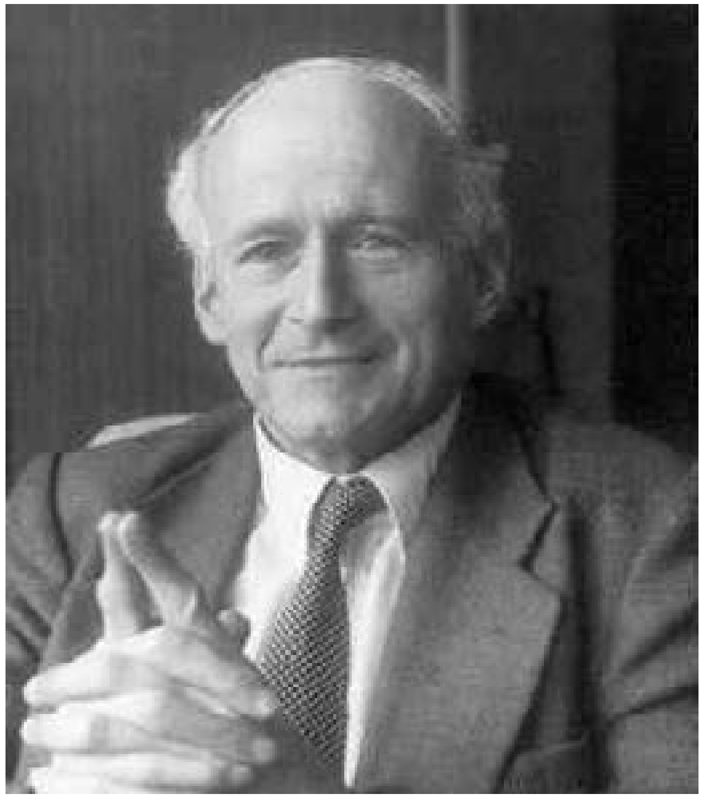
$$P_x = \lim_{N\to\infty} \frac{1}{2NT_0} E_x^{(N)} = \frac{1}{T_0}\int_{t=t_0}^{t_0+T_0} |x(t)|^2 \, \mathrm{d}t$$

The Dirac distribution

- A *distribution* generalizes the classical concept of a function
- Distributions are also known as *generalized functions*
- Distributions were introduced by the Russian mathematician Sergei Sobolev in his work on second-order hyperbolic partial differential equations (loosely speaking, differential equations that describe wave phenomena)
- Distribution theory was developed further and extended by the French mathematician Laurent Schwartz

The Dirac distribution

Sergey Sobolev
Born 1908
Died 1989



Laurent Schwartz
Born 1915
Died 2002

The Dirac distribution

- We only give a brief and informal introduction to distributions
- Much more on distribution theory can be found in
  - A.H. ZEMANIAN, Distribution Theory and Transform Analysis – An introduction to Generalized Functions, with Applications, Dover Publications, 2003
  - M.J. LIGHTHILL, An Introduction to Fourier Analysis and Generalised Functions, Cambridge Monographs on Mechanics, 2008

The Dirac distribution

- Before discussing distributions, we first introduce the space of *testing functions* $\mathscr{D}$
- The space of testing functions $\mathscr{D}$ consists of all complex-valued functions $\varphi(t)$ that are infinitely smooth and vanish outside some finite interval
- Infinitely smooth means: can be differentiated an infinite number of times

The Dirac distribution

- The finite interval (support) need not be the same for all testing functions*
- The support of testing function $\varphi_1(t)$ may be different from the support of testing function $\varphi_2(t)$
- Example of a testing function:

$$\varphi(t) = \begin{cases} \exp\left(\dfrac{1}{t^2-1}\right) & \text{for } |t| < 1 \\ 0 & \text{for } |t| \geq 1 \end{cases}$$

* Recall that the support of a function f is the set of points in the domain of f where f is nonzero

The Dirac distribution

- In general, a *functional* is a rule that assigns a number to every element of a certain set of functions
- We take the space of testing functions $\mathscr{D}$ as this set and consider functionals that assign a complex number to every element of $\mathscr{D}$
- In our case a functional is a rule that assigns a complex number to every testing function in $\mathscr{D}$
- The number that a functional f assigns to a testing function φ is denoted as $\langle f, \varphi \rangle$

The Dirac distribution

- Example: Let $f(t)$ be an integrable function
- By this we mean integrable over every finite interval
- Corresponding to this function, we can define a functional f through the integral

$$\langle f, \varphi \rangle = \langle f(t), \varphi(t) \rangle = \int_{t=-\infty}^{\infty} f(t)\varphi(t)\,\mathrm{d}t = \text{a number}$$

The Dirac distribution

- A *distribution* is a functional f with two additional properties

  * Linearity:

  $$\langle f, \varphi_1 + \varphi_2 \rangle = \langle f, \varphi_1 \rangle + \langle f, \varphi_2 \rangle$$

  for any two testing functions φ_1 and φ_2 from $\mathscr{D}$ and

  $$\langle f, \alpha\varphi \rangle = \alpha \langle f, \varphi \rangle$$

  for any complex number α

  * Continuity: For any set of testing functions $\{\varphi_n\}_{n=1}^{\infty}$ that converges in $\mathscr{D}$ to φ, the sequence of numbers $\{\langle f, \varphi_n \rangle\}_{n=1}^{\infty}$ converges to $\langle f, \varphi \rangle$

- A distribution is a continuous linear functional on the space of testing functions $\mathscr{D}$

The Dirac distribution

- Up to this point we have associated a distribution f to an ordinary function f
- For example, with the function $f(t) = p(t)$ we can associate the distribution

$$\langle p, \varphi \rangle = \int_{t=-\infty}^{\infty} p(t)\varphi(t)\,\mathrm{d}t = \int_{t=0}^{1} \varphi(t)\,\mathrm{d}t = \text{a number}$$

- Distributions associated with ordinary functions are called *regular*

The Dirac distribution

- Let us now consider a distribution that assigns the value $\varphi(0)$ to a testing function $\varphi(t)$
- This distribution is written as $\delta(t)$ and is called the *Dirac distribution*
- By definition, we have

$$\langle \delta(t), \varphi(t) \rangle = \varphi(0)$$

- In words: you take a testing function $\varphi(t)$ from $\mathscr{D}$. The Dirac distribution assigns the value $\varphi(0)$ to this testing function

The Dirac distribution

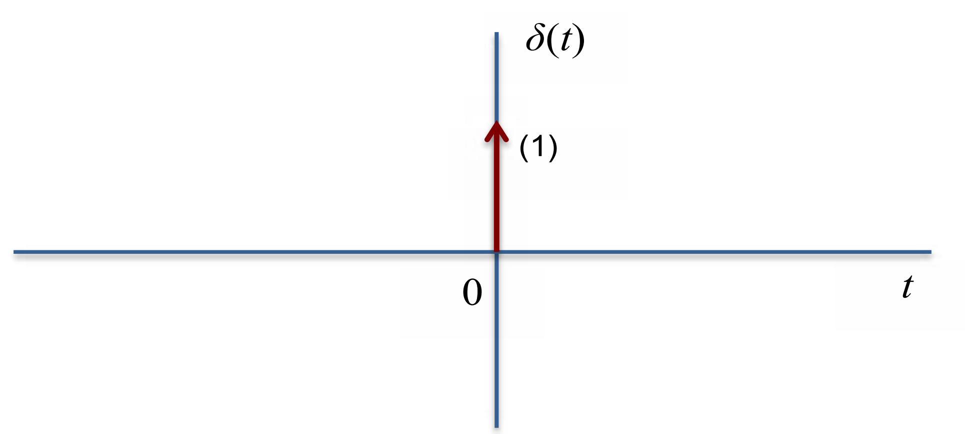
- Using the integral, we have

$$\langle \delta(t), \varphi(t) \rangle = \int_{t=-\infty}^{\infty} \delta(t)\varphi(t)\,\mathrm{d}t = \varphi(0)$$

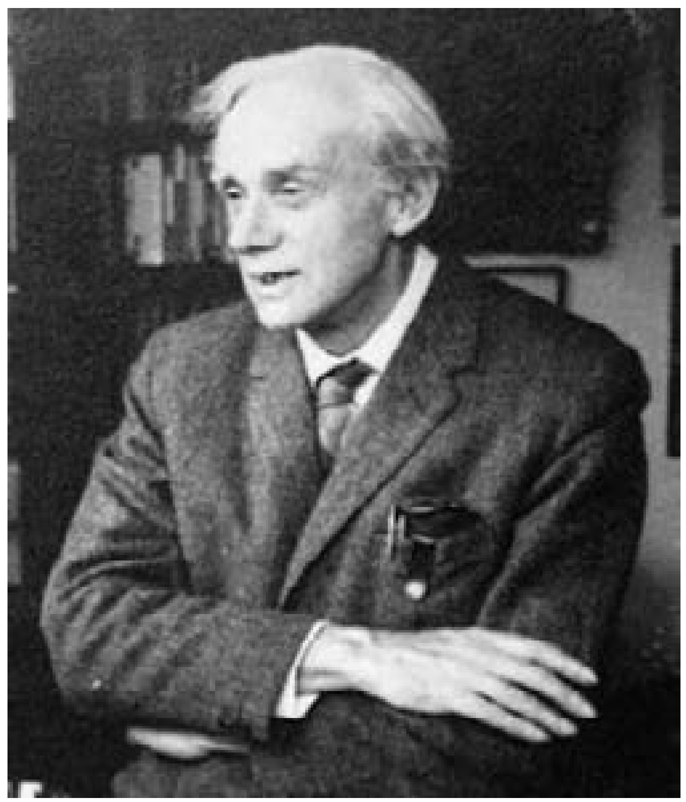
- No ordinary function has this property
- The Dirac distribution is an example of a *singular distribution function*

The Dirac distribution

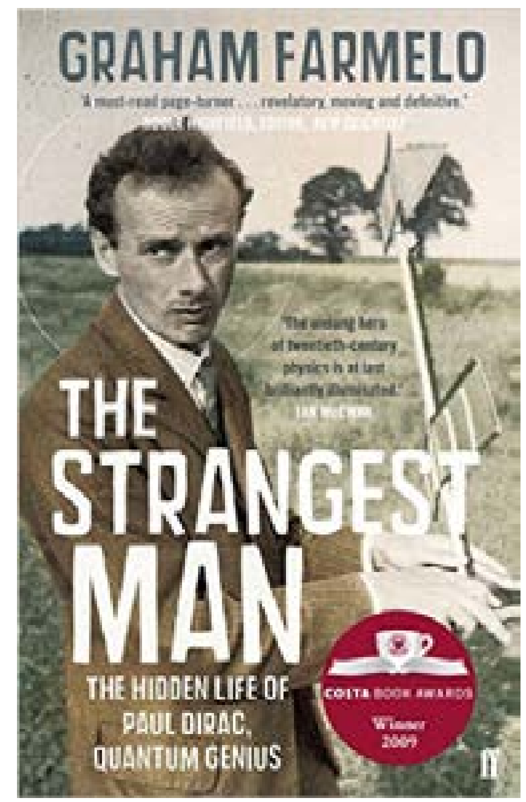
- We write $f(t) = \delta(t)$ only symbolically (as if the Dirac distribution is an ordinary function)
- The Dirac distribution is sometimes called the delta function, Dirac impulse function, or simply impulse function
- The Dirac distribution is called the "stoot" in Dutch



The Dirac distribution



Paul Dirac
Born 1902
Died 1984



Biography

The Dirac distribution

- Consider the ordinary Gaussian ($\epsilon > 0$)

$$f_\epsilon(t) = \frac{1}{\sqrt{\pi\epsilon}} e^{-t^2/\epsilon}$$

- This function is normalized in the sense that

$$\int_{t=-\infty}^{\infty} f_\epsilon(t)\,\mathrm{d}t = 1 \qquad \text{for any } \epsilon > 0$$

The Dirac distribution

- To this Gaussian function we can associate the regular distribution

$$\langle f_\epsilon(t), \varphi(t)\rangle = \int_{t=-\infty}^{\infty} f_\epsilon(t)\varphi(t)\,\mathrm{d}t = \frac{1}{\sqrt{\pi\epsilon}} \int_{t=-\infty}^{\infty} e^{-t^2/\epsilon}\varphi(t)\,\mathrm{d}t$$

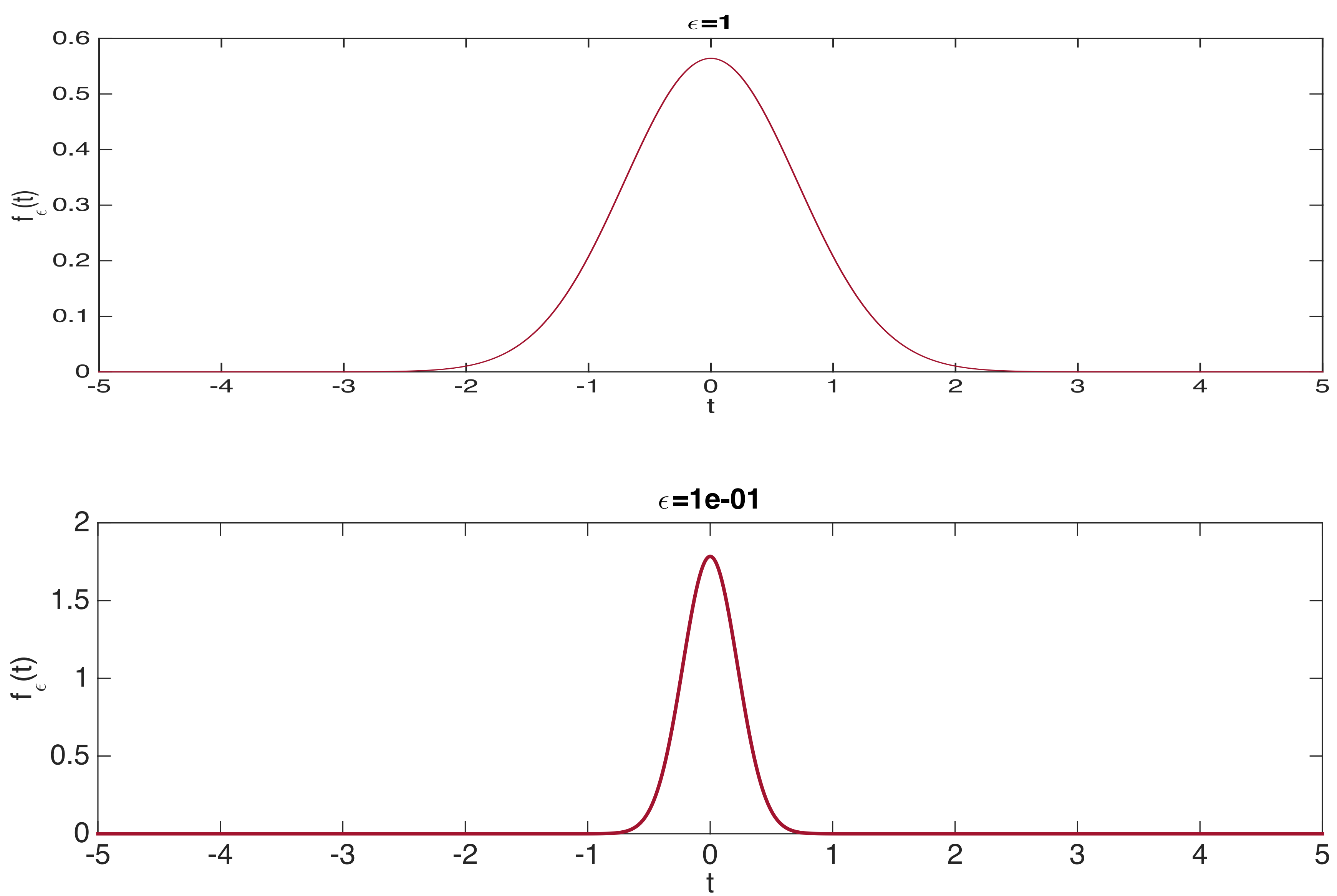
- For “very small" values of ϵ, we have

$$\langle f_\epsilon(t), \varphi(t)\rangle = \frac{1}{\sqrt{\pi\epsilon}} \int_{t=-\infty}^{\infty} e^{-t^2/\epsilon}\varphi(t)\,\mathrm{d}t \approx \varphi(0) \int_{t=-\infty}^{\infty} f_\epsilon(t)\,\mathrm{d}t = \varphi(0)$$

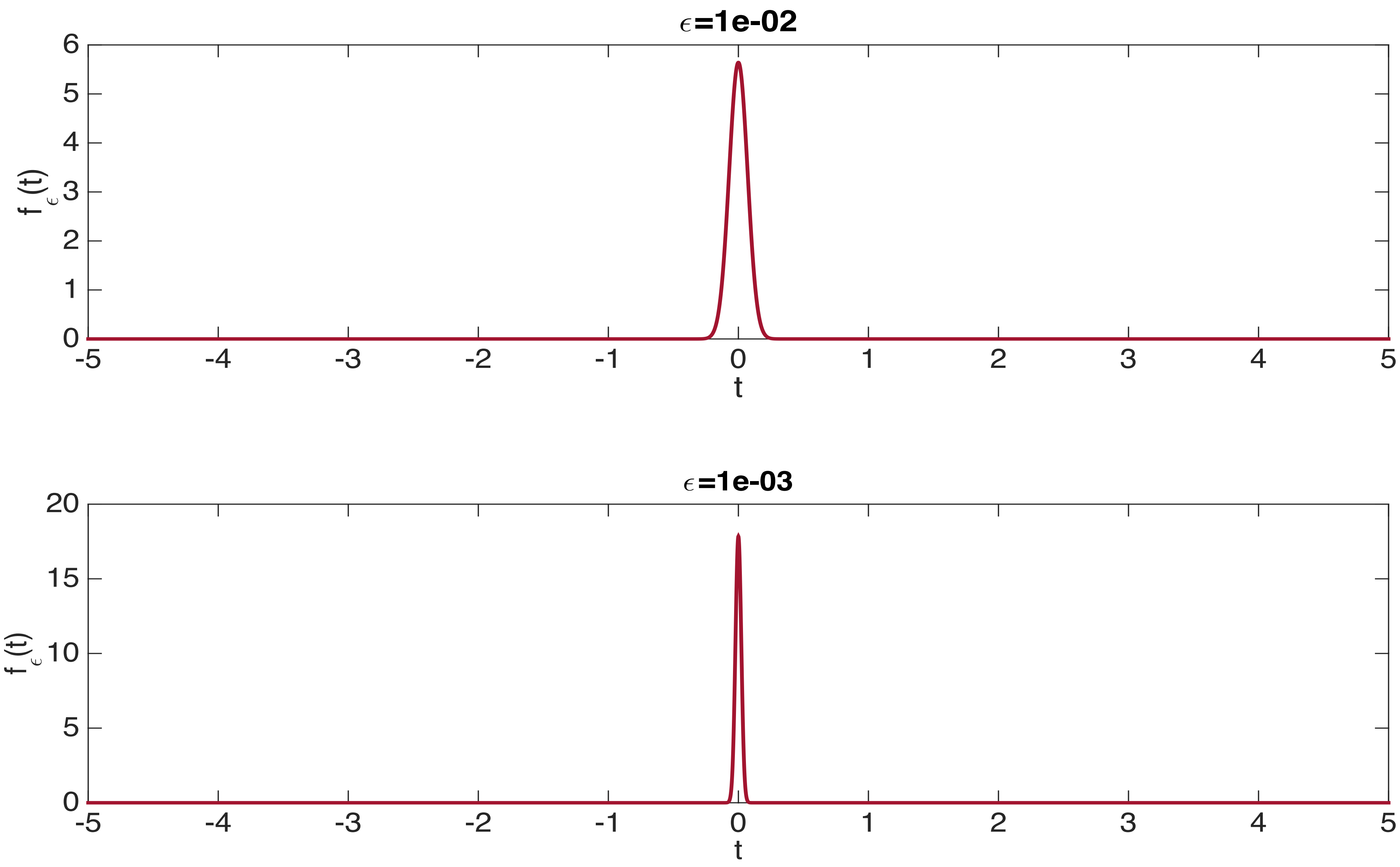
- and we may write

$$\delta(t) = \lim_{\epsilon\downarrow 0} f_\varepsilon(t) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{\pi\epsilon}} e^{-t^2/\epsilon}$$

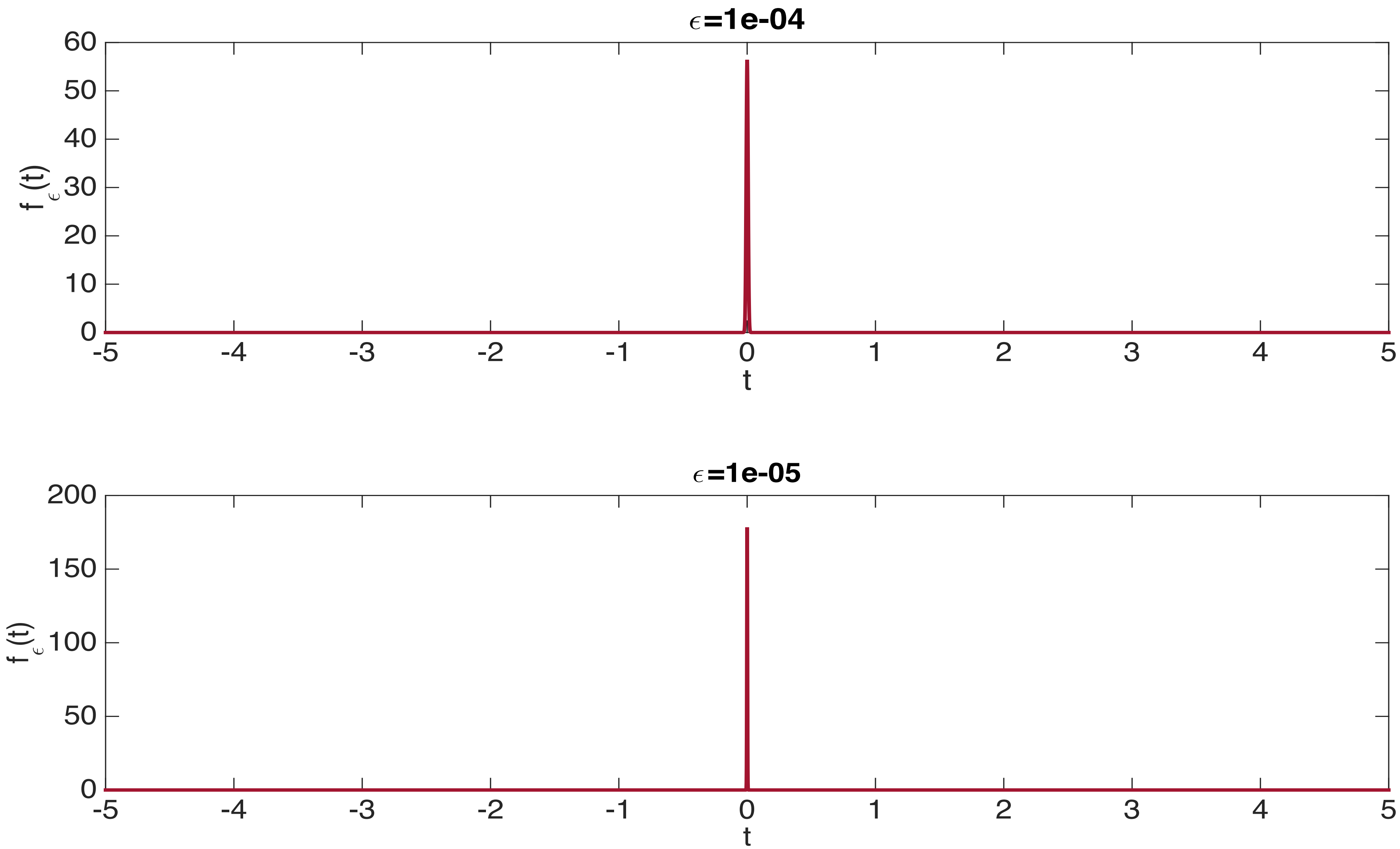
The Dirac distribution



The Dirac distribution



The Dirac distribution



The Dirac distribution

- Clearly, we have

$$\int_{t=a}^{b} \delta(t)\,\mathrm{d}t = \begin{cases} 1 & \text{if } 0 \in (a,b) \\ 0 & \text{if } 0 \notin (a,b) \end{cases}$$

- Also note that if t is expressed in seconds then the SI unit of the Dirac distribution is s^{-1}

- If t is expressed in meters (in this case we typically use the letter x instead of t) then the SI unit of the Dirac distribution is m^{-1}

The Dirac distribution

- Another one that does the same job is

$$\delta(t) = \lim_{a\to\infty} \frac{\sin(at)}{\pi t}$$

- Note that since

$$\frac{\sin(at)}{\pi t} = \frac{1}{2\pi}\int_{\Omega=-a}^{a} e^{\mathrm{j}\Omega t}\,\mathrm{d}\Omega$$

  we also have

$$\delta(t) = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\infty} e^{\mathrm{j}\Omega t}\,\mathrm{d}\Omega$$

- This latter relation is sometimes referred to as a *completeness relation*
- The above expressions only make sense as distributions, of course

The Dirac distribution

- Given a signal $f(t)$ continuous at the origin
- We claim that the two distributions $f(t)\delta(t)$ and $f(0)\delta(t)$ are equal
- In other words

$$f(t)\delta(t) = f(0)\delta(t)$$

- We show this by working out the distributions on the left- and right-hand side
- Both distributions should produce the same number
- Let's start with the left-hand side

The Dirac distribution

- We have

$$\begin{aligned}\langle f(t)\delta(t), \varphi(t)\rangle &= \int_{t=-\infty}^{\infty} \left[f(t)\delta(t)\right]\varphi(t)\,\mathrm{d}t \\ &= \int_{t=-\infty}^{\infty} \delta(t)\left[f(t)\varphi(t)\right]\mathrm{d}t \\ &= f(0)\varphi(0)\end{aligned}$$

The Dirac distribution

- Now for the right-hand side:

$$\begin{aligned}\langle f(0)\delta(t), \varphi(t)\rangle &= \int_{t=-\infty}^{\infty} \left[f(0)\delta(t)\right]\varphi(t)\,\mathrm{d}t \\ &= f(0)\int_{t=-\infty}^{\infty} \delta(t)\varphi(t)\,\mathrm{d}t \\ &= f(0)\varphi(0)\end{aligned}$$

- We conclude that our claim is correct

The Dirac distribution

- The shifted delta distribution $\delta(t-t_0)$ associates the number $\varphi(t_0)$ to a testing function $\varphi(t)$
- We have

$$\langle \delta(t-t_0), \varphi(t) \rangle = \varphi(t_0)$$

- or in terms of an integral

$$\int_{t=-\infty}^{\infty} \delta(t-t_0)\varphi(t)\, \mathrm{d}t = \varphi(t_0)$$

- This is sometimes called the *sifting property* of the Dirac distribution (zeefeigenschap van de Dirac distributie)

The Dirac distribution

- For the shifted Dirac distribution we have

$$\int_{t=a}^{b} \delta(t-t_0)\,\mathrm{d}t = \begin{cases} 1 & \text{if } t_0 \in (a,b) \\ 0 & \text{if } t_0 \notin (a,b) \end{cases}$$

- Moreover, for a signal $f(t)$ continuous at $t = t_0$

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

The Dirac distribution - scaling property

- Let a be a nonzero real number
- Scaling property of the Dirac distribution

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

- We verify that these distributions are indeed equal
- For the left-hand side we have

$$\langle \delta(at), \varphi(t) \rangle = \int_{t=-\infty}^{\infty} \delta(at)\varphi(t)\,\mathrm{d}t$$

The Dirac distribution - scaling property

- First consider the case $a > 0$. Setting $\tau = at$ results in

$$\langle \delta(at), \varphi(t) \rangle = \frac{1}{a} \int_{\tau=-\infty}^{\infty} \delta(\tau)\varphi(\tau/a)\,\mathrm{d}\tau = \frac{1}{a}\varphi(0)$$

- Next, consider the case $a < 0$. With $\tau = at$ we now have

$$\begin{aligned} \langle \delta(at), \varphi(t) \rangle &= \frac{1}{a} \int_{\tau=\infty}^{-\infty} \delta(\tau)\varphi(\tau/a)\,\mathrm{d}\tau \\ &= -\frac{1}{a} \int_{\tau=-\infty}^{\infty} \delta(\tau)\varphi(\tau/a)\,\mathrm{d}\tau \\ &= -\frac{1}{a}\varphi(0) \end{aligned}$$

The Dirac distribution - scaling property

- Both cases ($a > 0$ and $a < 0$) can be combined as

$$\langle \delta(at), \varphi(t) \rangle = \frac{1}{|a|}\varphi(0)$$

- For the right-hand side we obtain

$$\begin{aligned}
\langle \frac{1}{|a|}\delta(t), \varphi(t) \rangle &= \int_{t=-\infty}^{\infty} \frac{1}{|a|}\delta(t)\varphi(t)\,\mathrm{d}t \\
&= \frac{1}{|a|}\int_{t=-\infty}^{\infty} \delta(t)\varphi(t)\,\mathrm{d}t \\
&= \frac{1}{|a|}\varphi(0)
\end{aligned}$$

- and we conclude that the two distributions are equal

The Dirac distribution is even

- Special case: $a = -1$. We obtain

$$\delta(-t) = \delta(t)$$

- The Dirac distribution is even

The Dirac distribution - derivative of the unit step function

- Following the same verification procedure as above, we can also show that

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \delta(t)$$

- The derivative of the unit step function is equal to the Dirac distribution!
- The unit step function is not differentiable at $t = 0$, of course, but it can be differentiated in the sense of distributions

The Dirac distribution - derivative of the unit step function

- We start with the left-hand side

$$\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \varphi(t) \rangle = \int_{t=-\infty}^{\infty} \frac{\mathrm{d}u}{\mathrm{d}t} \varphi(t)\,\mathrm{d}t = \lim_{T\to\infty} u(t)\,\varphi(t)\Big|_{t=-T}^{T} - \int_{t=-\infty}^{\infty} u(t) \frac{\mathrm{d}\varphi}{\mathrm{d}t}\,\mathrm{d}t$$

- The first-term on the right-hand side of the above equation vanishes, since a testing function has bounded support

The Dirac distribution - derivative of the unit step function

- We are left with

$$\langle \frac{\mathrm{d}u}{\mathrm{d}t}, \varphi(t) \rangle = -\int_{t=-\infty}^{\infty} u(t) \frac{\mathrm{d}\varphi}{\mathrm{d}t} \,\mathrm{d}t = -\int_{t=0}^{\infty} \frac{\mathrm{d}\varphi}{\mathrm{d}t} \,\mathrm{d}t = \varphi(0) - \lim_{T \to \infty} \varphi(T) = \varphi(0)$$

- For the right-hand side we have (by definition)

$$\langle \delta(t), \varphi(t) \rangle = \varphi(0)$$

- and we conclude once again that the two given distributions are equal

- The derivative of the Dirac distribution, denoted as $\delta'(t)$, is defined as the distribution that assigns the value $-\varphi'(0)$ to a testing function φ belonging to $\mathscr{D}$:

$$\langle \delta'(t), \varphi(t) \rangle = -\varphi'(0)$$

- Here, we use a prime to indicate a derivative (this is more or less standard notation)

- Perhaps you are wondering why the derivative of $\delta(t)$ is defined as above

- The reason is that we can now again manipulate with the Dirac distribution as if it is an ordinary function

$$\begin{aligned}\langle \delta'(t), \varphi(t) \rangle &= \int_{t=-\infty}^{\infty} \frac{\mathrm{d}\delta(t)}{\mathrm{d}t} \varphi(t) \,\mathrm{d}t \\ &= \lim_{T\to\infty} \delta(t)\,\varphi(t)\Big|_{t=-T}^{T} - \int_{t=-\infty}^{\infty} \delta(t) \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} \,\mathrm{d}t = -\varphi'(0)\end{aligned}$$

The Dirac distribution - derivative of the Dirac distribution

- The shifted derivative of the Dirac distribution is defined as

$$\langle \delta'(t-t_0), \varphi(t) \rangle = -\varphi'(t_0)$$

- Let f be a signal continuously differentiable at $t=0$
- We have

$$f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$$

- We verify the above statement

- Left-hand side

$$\begin{aligned}
\langle f(t)\delta'(t), \varphi(t)\rangle &= \int_{t=-\infty}^{\infty} [f(t)\delta'(t)]\varphi(t)\,\mathrm{d}t = \int_{t=-\infty}^{\infty} \delta'(t)[f(t)\varphi(t)]\,\mathrm{d}t \\
&= \lim_{T\to\infty} \delta(t)\, f(t)\, \varphi(t)\Big|_{t=-T}^{T} - \int_{t=-\infty}^{\infty} \delta(t)\frac{\mathrm{d}}{\mathrm{d}t}\Big[f(t)\varphi(t)\Big]\,\mathrm{d}t \\
&= -\int_{t=-\infty}^{\infty} \delta(t)\Big[f'(t)\varphi(t) + f(t)\varphi'(t)\Big]\,\mathrm{d}t \\
&= -f'(0)\varphi(0) - f(0)\varphi'(0)
\end{aligned}$$

The Dirac distribution - derivative of the Dirac distribution

- Right-hand side

$$\begin{aligned}\langle -f'(0)\delta(t) + f(0)\delta'(t), \varphi(t)\rangle &= -f'(0)\langle \delta(t), \varphi(t)\rangle + f(0)\langle \delta'(t), \varphi(t)\rangle \\ &= -f'(0)\varphi(0) - f(0)\varphi'(0)\end{aligned}$$

- Conclusion: the two distributions are equal

The Dirac distribution - summary

Summary: Properties of the Dirac distribution	
$\int_{t=a}^{b} \delta(t - t_0)\,\mathrm{d}t = \begin{cases} 1 & \text{if } t_0 \in (a,b) \\ 0 & \text{if } t_0 \notin (a,b) \end{cases}$	integration property
$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$	$f(t)$ continuous at $t = t_0$
$\delta(at) = \frac{1}{\lvert a \rvert}\delta(t)$	$a \in \mathbb{R} \setminus \{0\}$, scaling property
$\frac{\mathrm{d}u}{\mathrm{d}t} = \delta(t)$	derivative of the unit step function
$f(t)\delta'(t) = -f'(0)\delta(t) + f(0)\delta'(t)$	$f(t)$ continuously differentiable at $t = 0$

Exercise 1.1 Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{sign}(t) = 2\delta(t)$$

Exercise 1.2 Show that

$$\delta(at+b) = \frac{1}{|a|}\delta(t+b/a) \qquad a \neq 0$$

Exercise 1.3 Determine

$$\int_{\tau=-\infty}^{t} \delta(\tau)\,\mathrm{d}\tau$$

Answer: $u(t)$

The Dirac distribution - exercises

Exercise 1.4 Determine

$$\int_{\tau=-\infty}^{t} \delta(\tau) f(\tau)\, \mathrm{d}\tau$$

Answer: $f(0)\,u(t)$

Exercise 1.5 Compute

$$\int_{t=-\infty}^{\infty} \delta(t) f(t-t_0)\, \mathrm{d}t.$$

Answer: $f(-t_0)$

Exercise 1.6 Compute

$$\int_{t=-\infty}^{\infty} \delta(t)\, t\, \mathrm{d}t.$$

Answer: 0

The Dirac distribution - exercises

Exercise 1.7 Sketch the signal

$$f(t) = \sin(\pi t)u(t)$$

and compute $f'(t)$.

Exercise 1.8 Sketch the signal

$$g(t) = \cos(\pi t)u(t)$$

and compute $g'(t)$.

Exercise 1.9 Compute $p'(t)$, where $p(t)$ is the rectangular pulse function. Sketch $p(t)$ and $p'(t)$.

Exercise 1.10 Compute $\Lambda'(t)$, where $\Lambda(t)$ is the triangular pulse function. Sketch $\Lambda(t)$ and $\Lambda'(t)$.

Exercise 1.11 Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}|t| = \mathrm{sign}(t)$$

Exercise 1.12 Plot the signal

$$f(t) = \mathrm{sign}(t) - \mathrm{sign}(t-1)$$

Exercise 1.13 Show that

$$t\delta'(t) = -\delta(t)$$

Exercise 1.14 Explain why

$$\int_{t=-\infty}^{\infty} \delta'(t)\,\mathrm{d}t = 0$$