17. Exercises for part 2 (second part)

1. Convolution
2. Realizations
3. Filter design (analog, digital)

Exercises

Convolution

Given a signal $x[n]$ with samples $x = [\cdots, 0, 0, \boxed{1}, 2, 3, 2, 1, 0, 0, \cdots]$ (the square indicates $x[0]$), and a filter $h[n]$ with impulse response $h = [\cdots, 0, 0, \boxed{1}, 1, 0, 0, \cdots]$

1. Determine the convolution $y = x * h$.
2. Determine from the time series the z-transforms $X(z)$, $H(z)$ and $Y(z)$.
3. Show that $Y(z) = X(z)H(z)$
4. If x has length N_x (i.e., the interval of non-zero samples) and h length N_h, how many samples has y?

Exercises

Answer:

1. Use the definition $y[n] = \sum_k h[k]x(n-k)$, here $y[n] = x[n] + x[n-1]$.

$$
\begin{array}{llllllll}
x[n]: & [\boxed{1} & 2, & 3, & 2, & 1, & 0, & 0\cdots] \\
x[n-1]: & [\boxed{0} & 1 & 2, & 3, & 2, & 1, & 0\cdots] \\
y[n]: & [\boxed{1} & 3, & 5, & 5, & 3, & 1, & 0\cdots]
\end{array}
$$

2.

$$
\begin{aligned}
X(z) &= 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} \\
H(z) &= 1 + z^{-1} \\
Y(z) &= 1 + 3z^{-1} + 5z^{-2} + 5z^{-3} + 3z^{-4} + z^{-5}
\end{aligned}
$$

3. $H(z)X(z) = (1 + z^{-1})(1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4})$

   Computing the product, we see that the same calculations are done as in item 1.

4. From the construction of the convolution it follows $N_y = N_x + N_h - 1$.

Exercises

Frequency domain

Given a filter $h_1[n]$ with impulse response $h_1 = [\cdots, 0, 0, \boxed{1}, 1, 0, 0, \cdots]$, and a filter $h_2[n]$ with impulse response $h_2 = [\cdots, 0, 0, \boxed{1}, 0, 1, 0, 0, \cdots]$.

1. Determine $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$. Show that these are linear-phase filters.
2. What are the amplitude responses $|H_1(e^{j\omega})|^2$ and $|H_2(e^{j\omega})|^2$? Make drawings.
3. Suppose $h_3 = h_1 * h_2$. Determine the impulse response $h_3[n]$, and $|H_3(e^{j\omega})|^2$. Make a drawing.
4. Show that $|H_3(e^{j\omega})|^2 = |H_1(e^{j\omega})|^2 |H_2(e^{j\omega})|^2$.

Exercises

Answer:

1.

$$
\begin{aligned}
H_1(z) &= 1 + z^{-1} \\
H_1(e^{j\omega}) &= 1 + e^{-j\omega} = e^{-j\omega/2}(e^{j\omega/2} + e^{-j\omega/2}) = 2e^{-j\omega/2}\cos(\omega/2)
\end{aligned}
$$

$$
\begin{aligned}
H_2(z) &= 1 + z^{-2} \\
H_2(e^{j\omega}) &= 1 + e^{-j2\omega} = e^{-j\omega}(e^{j\omega} + e^{-j\omega}) = 2e^{-j\omega}\cos(\omega)
\end{aligned}
$$

2.

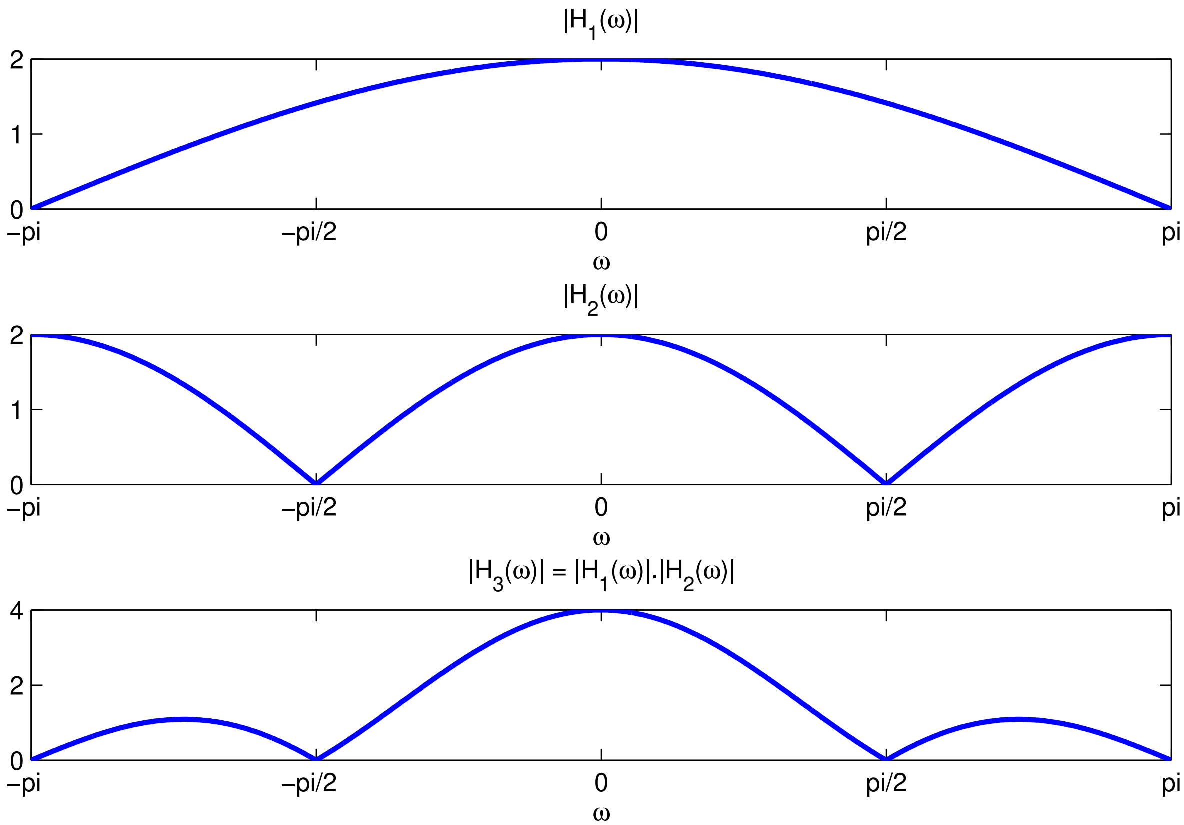
$$
\begin{aligned}
|H_1(e^{j\omega})|^2 &= (1 + e^{-j\omega})(1 + e^{j\omega}) = 2 + e^{-j\omega} + e^{j\omega} = 2 + 2\cos(\omega) \\
|H_2(e^{j\omega})|^2 &= (1 + e^{-j2\omega})(1 + e^{j2\omega}) = 2 + e^{-j2\omega} + e^{j2\omega} = 2 + 2\cos(2\omega)
\end{aligned}
$$

3.

$$
\begin{aligned}
h_3 &= [\cdots, 0, 0, \boxed{1}, 1, 1, 1, 0, 0, \cdots] \\
H_3(z) &= 1 + z^{-1} + z^{-2} + z^{-3} = \tfrac{1-z^{-4}}{1-z^{-1}} \\
H_3(e^{j\omega}) &= \tfrac{1-e^{-4j\omega}}{1-e^{-j\omega}} = \tfrac{e^{-j2\omega}}{e^{-j\omega/2}} \tfrac{e^{2j\omega}-e^{-2j\omega}}{e^{j\omega/2}-e^{-j\omega/2}} = e^{-j\frac{3}{2}\omega}\frac{\sin(2\omega)}{\sin(\omega/2)}
\end{aligned}
$$

4. Algebraically this is quite involved but it is simple to verify in matlab.

Exercises



|H1(ω)|
2
1
0
−pi
−pi/2
0
pi/2
pi
ω
|H2(ω)|
2
1
0
−pi
−pi/2
0
pi/2
pi
ω
|H3(ω)| = |H1(ω)|.|H2(ω)|
4
2
0
−pi
−pi/2
0
pi/2
pi
ω

Exercises

Realizations

Given the transfer function of a causal system: $H(z) = b_0 \frac{1 + bz^{-1}}{1 + az^{-1}}$

a. Draw the Direct Form I and II realizations. What is the corresponding difference equation?

b. Draw the pole/zero diagram for $a = 0.5$ and $b = -0.6$.

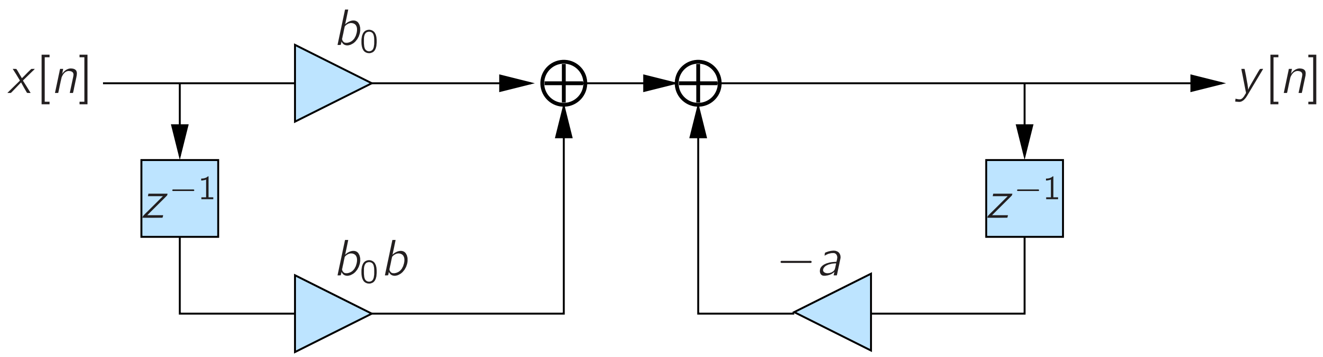
c. What is the ROC?

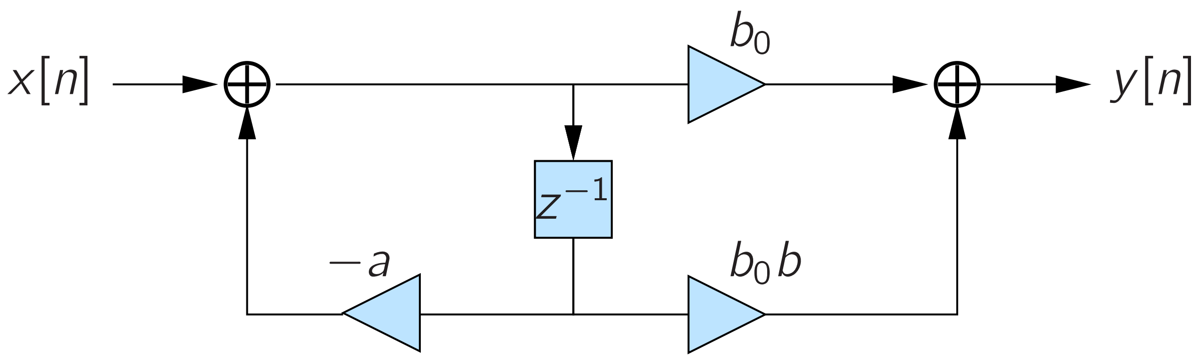
d. Is this a stable system, and why?

Answer:

$$H(z) = b_0 \frac{1 + bz^{-1}}{1 + az^{-1}}$$

a. Draw the Direct Form I and II realizations.



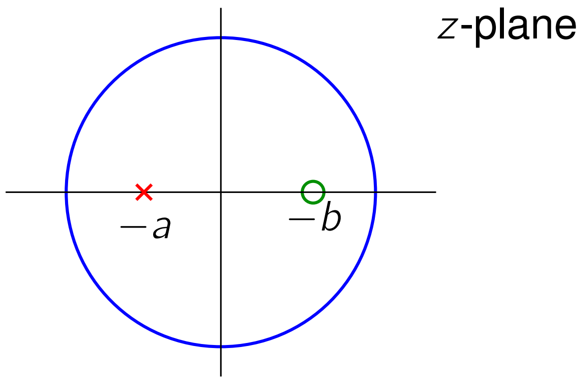


Difference equation: $y[n] + ay[n-1] = b_0 x[n] + b_0 b_1 x[n-1]$

Exercises

b. Draw the pole/zero diagram for $a = 0.5$ and $b = -0.6$

Pole: $z = -a$; zero: $z = -b$.



c. What is the ROC? $\{|z| > 0.5\}$.

d. Is this a stable system? Yes, the unit circle is in the ROC.

(Equivalent: the system is causal and the poles are within the unit circle.)

Exercises

Filterdesign

What is the minimal filter order for an analog lowpass filter with specifications:

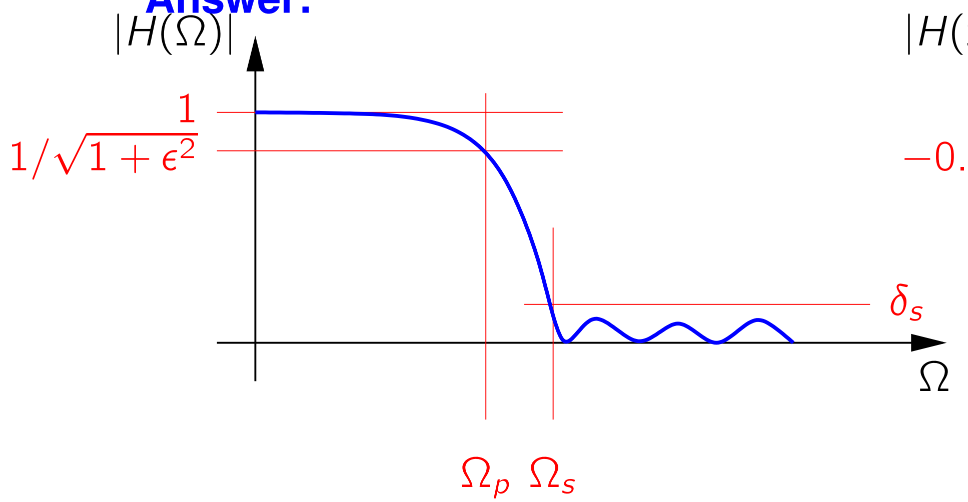
pass-band until 1.2 kHz, maximal ripple in the pass-band 0.5 dB

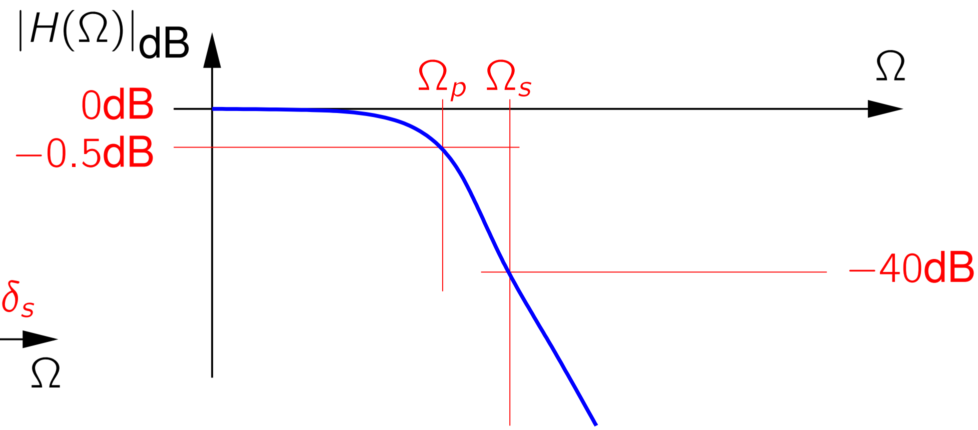
stop-band from 2.0 kHz, minimal damping in the stop-band 40 dB

a. for a Butterworth filter

b. for a Chebyshev filter

Answer:





Butterworth: $|H(\Omega)|^2 = \dfrac{1}{1+\epsilon^2(\Omega/\Omega_p)^{2N}}$

- Pass-band: $\dfrac{1}{1+\epsilon^2} = 10^{-0.5/10} = 0.89 \quad \Rightarrow \quad \epsilon = \sqrt{\dfrac{1}{0.89} - 1} = 0.35$
- Stop-band: $\delta_s = 10^{-40/20} = 0.01$
- For the filter order:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\Omega_s/\Omega_p)^{2N}} = \delta_s^2 \quad \Rightarrow \quad \left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{\frac{1}{\delta_s^2} - 1}{\epsilon^2} =: \frac{\delta^2}{\epsilon^2} \quad \Rightarrow \quad N = \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)}$$

Substitution gives $\delta = 99.995$ and $N = 11.1$, i.e., the filter order is $N \geq 12$

Exercises

Chebyshev:

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega/\Omega_p)} \quad \text{met} \quad \begin{cases} T_N(x) = \cos(N\cos^{-1}x) & \text{for } |x| \le 1 \\ T_N(x) = \cosh(N\cosh^{-1}x) & \text{for } |x| > 1 \end{cases}$$

Pass-band criterion results in the same $\epsilon = 0.35$ as for Butterworth

Stop-band criterion:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\Omega_s/\Omega_p)} = \delta_s^2 \quad \Rightarrow \quad T_N^2(\Omega_s/\Omega_p) = \frac{\frac{1}{\delta_s^2}-1}{\epsilon^2} =: \frac{\delta^2}{\epsilon^2}$$

$$\Rightarrow \quad \cosh(N\cosh^{-1}(\Omega_s/\Omega_p)) = \frac{\delta}{\epsilon} \quad \Rightarrow \quad N = \frac{\cosh^{-1}(\delta/\epsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

Substitution gives $N = 5.78$, hence the filter order is $N \geq 6$.

Note: in case $\cosh^{-1}(x)$ is not on your calculator, you can compute it as $\cosh^{-1}(x) = \ln(x+\sqrt{x^2-1})$. Derivation:

$$x = \cosh(y) := \frac{e^y + e^{-y}}{2}, \quad \sinh(y) := \frac{e^y - e^{-y}}{2}, \quad \cosh^2(y) - \sinh^2(y) = 1$$

Hence $\sinh(y) = \sqrt{x^2-1}$ and $e^y = \cosh(y) + \sinh(y) = x + \sqrt{x^2-1}$.

Exercises

Digital filter design

Use the bilinear transform to design a digital low-pass filter $H(z)$ with the following parameters:

> Pass-band: $0 \leq |\omega| \leq 0.3\pi$, maximal ripple 1 dB
>
> Stop-band: $0.35\pi \leq |\omega| \leq \pi$, minimal damping 60 dB

a Translate these specifications to the analog frequency domain

b What filter order is required if we use a Butterworth filter?

(To determine the filter coefficients, a computer is needed.)

Answer

a The bilinear transform results in $\Omega = \tan(\frac{\omega}{2})$. Here:

$$\Omega_p = \tan(0.3\pi/2) = 0.5095, \qquad \Omega_s = \tan(0.35\pi/2) = 0.6128$$

b The equation for a Butterworth filter is

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2(\Omega/\Omega_p)^{2N}}$$

We obtain

$$\text{Pass-band ripple:} \quad \frac{1}{1+\epsilon^2} = 10^{-1/10} \quad \Rightarrow \quad \epsilon = 0.5088$$

$$\text{Stop-band damping:} \quad \frac{1}{1 + \epsilon^2(\Omega_s/\Omega_p)^{2N}} \leq \delta_s^2 = 10^{-60/10} = 10^{-6}$$

Define $\delta := \sqrt{\frac{1}{\delta_s^2} - 1} = 999.9$ and simplify:

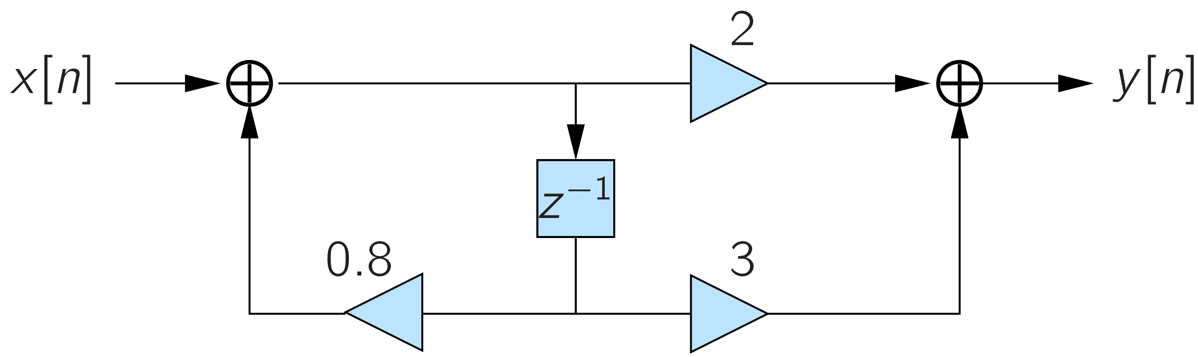
$$N \geq \frac{\log(\delta/\epsilon)}{\log(\Omega_s/\Omega_p)} = 41.07$$

This reults in $N = 42$.

Exercises

Realizations

Given the realization:

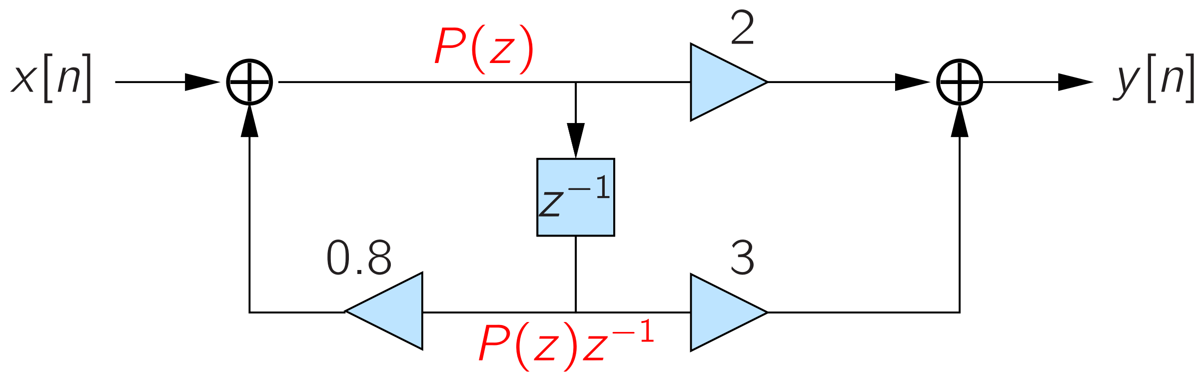


a. Determine the impulse response

b. Determine a realization for the inverse system: $y[n] \rightarrow x[n]$

Exercises

Introduce the extra parameter $P(z)$ (subsequently eliminated):



$$\begin{cases} P(z) = X(z) + 0.8P(z)z^{-1} \\ Y(z) = 2P(z) + 3P(z)z^{-1} \end{cases} \Rightarrow \begin{cases} P(z) = X(z)\frac{1}{1-0.8z^{-1}} \\ Y(z) = X(z)\frac{2+3z^{-1}}{1-0.8z^{-1}} \end{cases}$$

$$H(z) = \frac{2+3z^{-1}}{1-0.8z^{-1}} = \frac{2}{1-0.8z^{-1}} + z^{-1}\frac{3}{1-0.8z^{-1}} \Rightarrow h[n] = 2(0.8)^n u[n] + 3(0.8)^{n-1}u[n-1]$$

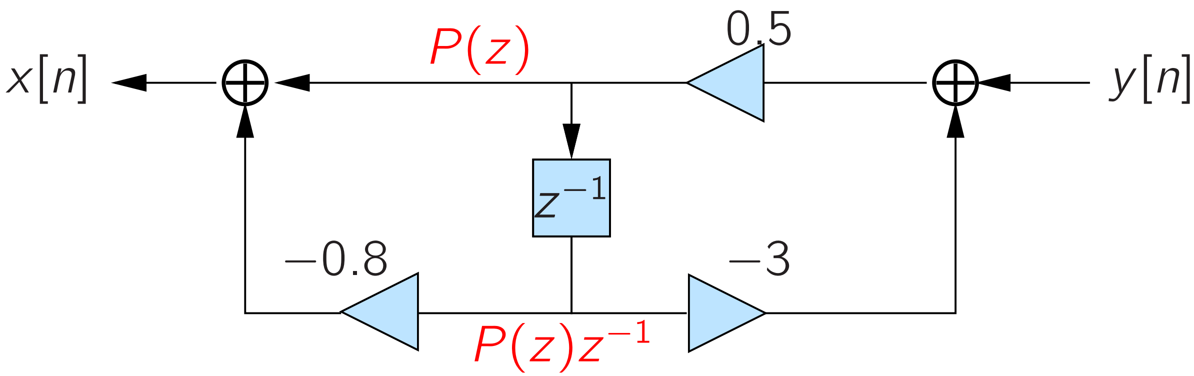
with $u[n]$ a unit step function.

Inverse system:

$$\begin{cases} P(z) = X(z) + 0.8P(z)z^{-1} \\ Y(z) = 2P(z) + 3P(z)z^{-1} \end{cases} \Rightarrow \begin{cases} P(z) = \frac{1}{2}(Y(z) - 3P(z)z^{-1}) \\ X(z) = P(z) - 0.8P(z)z^{-1} \end{cases}$$

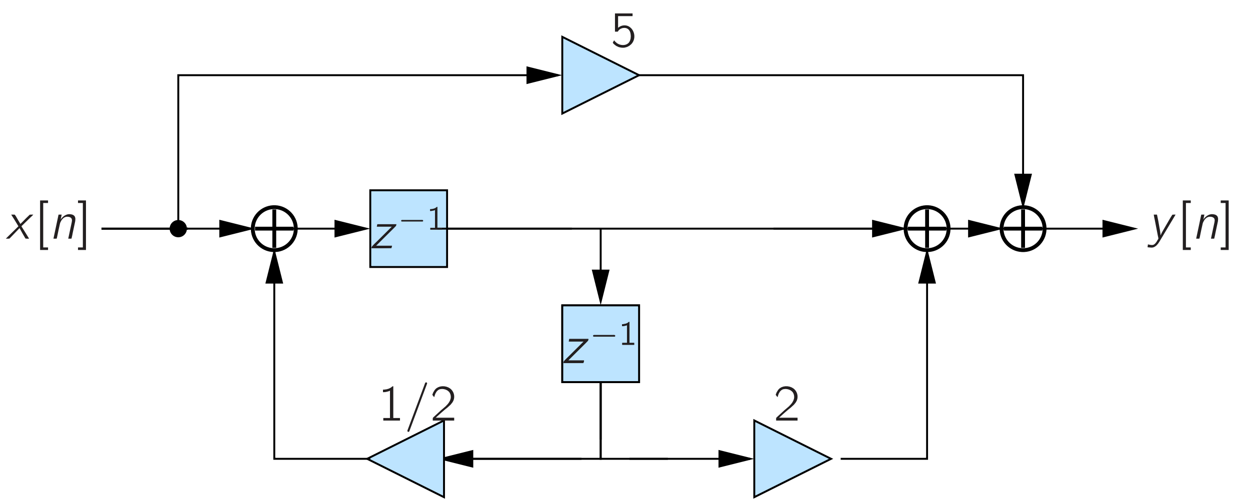
Exercises

$$\begin{cases} P(z) = \frac{1}{2}(Y(z) - 3P(z)z^{-1}) \\ X(z) = P(z) - 0.8P(z)z^{-1} \end{cases}$$



Exercises

a Determine the transfer function $H(z)$ of the following system:



b Draw the “Direct form II” realization

c Draw the transposed system.

d What is the transfer function of the transposed system?

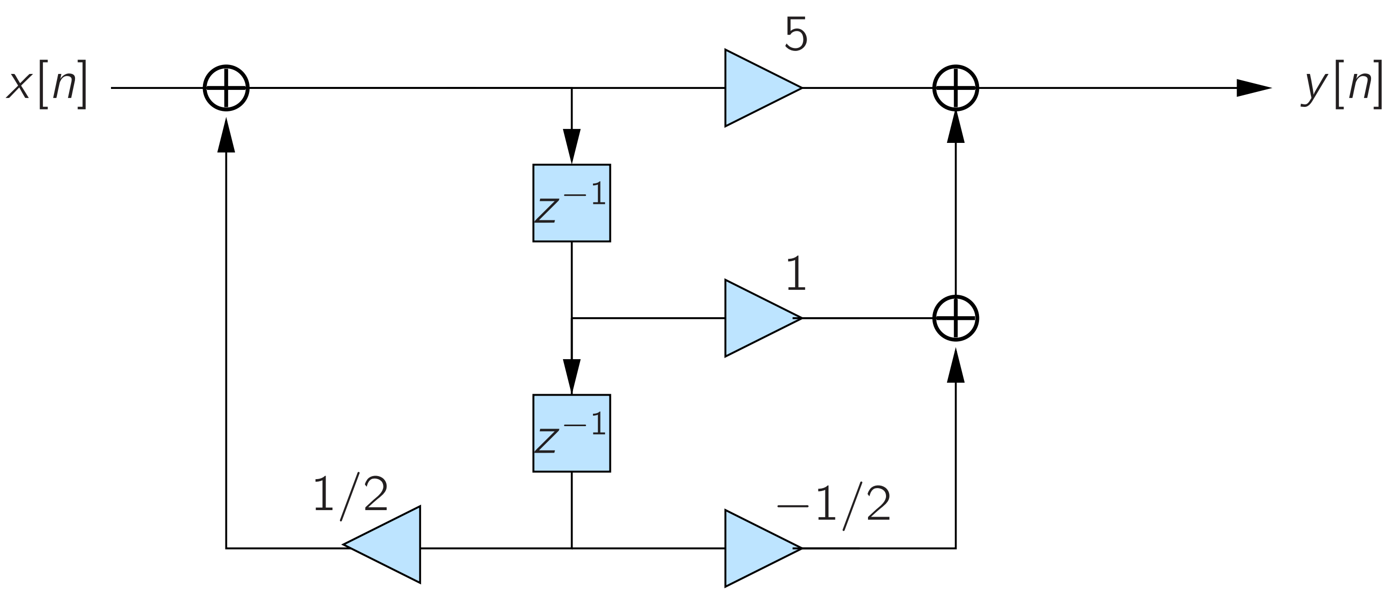
Exercises

Answer

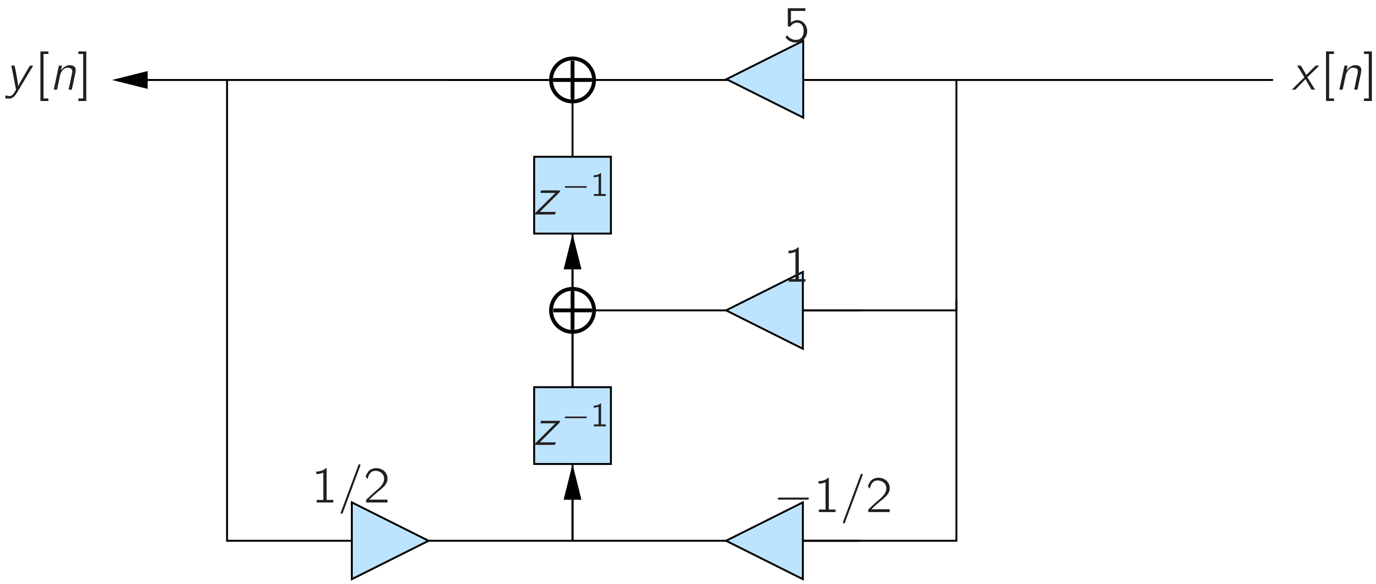
a

$$H(z) = 5 + \frac{z^{-1} + 2z^{-2}}{1 - 1/2z^{-2}} = \frac{5 + z^{-1} - 1/2z^{-2}}{1 - 1/2z^{-2}}$$

b Draw the “Directe form II” realization



c Draw the transposed system.



d What is the transfer function of the transposed system? Unchanged.

Exercises

We would like to design an analog *high-pass* filter with the following specifications:

– Pass-band: from $f_p = 50$ Hz; ripple in the pass-band: $\leq$ 1 dB

– Stop-band: until $f_s = 40$ Hz; stop-band damping: $\geq$ 30 dB.

We start with a Butterworth low-pass filter structure with the form

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_c)^{2N}}$$

a Make a neat drawing of $|H(\Omega)|^2$. Also indicate Ω_c, and the corresponding value of $|H(\Omega)|^2$.

We apply to $H(\Omega)$ a low-to-high frequency transform: $\Omega \rightarrow \Omega_c^2/\Omega$. Put $G(\Omega) = H(\Omega_c^2/\Omega)$.

b Give an expression for $|G(\Omega)|^2$?

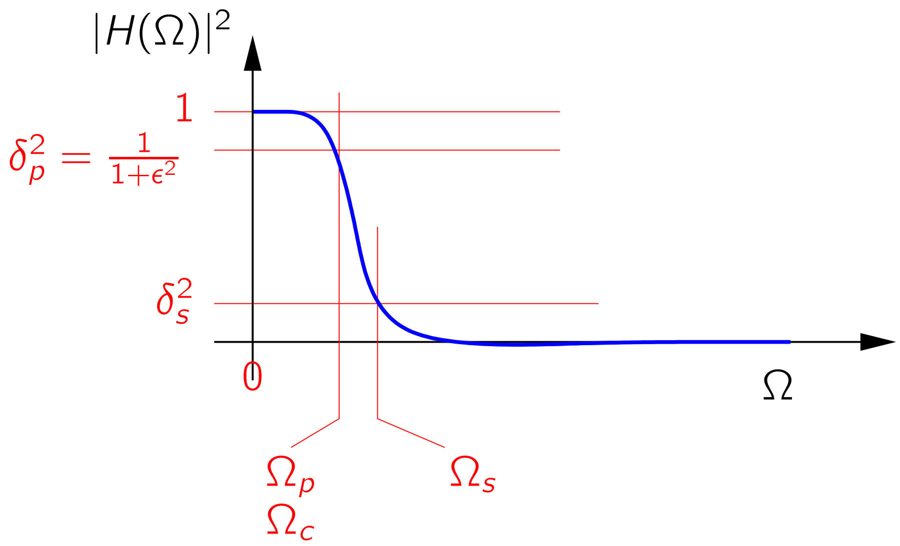
c Make a neat drawing of $|G(\Omega)|^2$. Also indicate Ω_c.

d What do you choose for Ω_c? Which value do you choose for $|G(\Omega_c)|^2$?

e Compute step-by-step ϵ and the required filter order N for $G(\Omega)$ such that the given specifications are met.

Answer

a

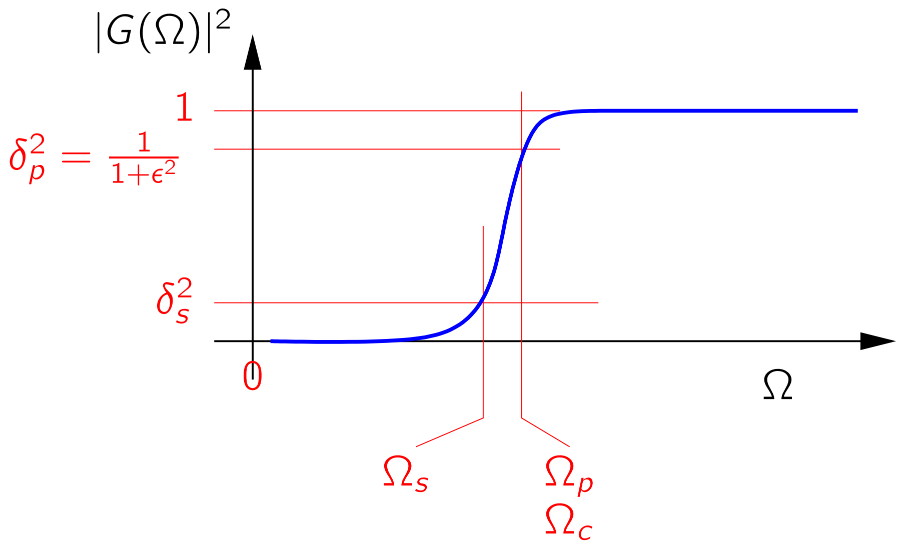


b

$$|G(\Omega)|^2 = \frac{1}{1 + \epsilon^2(\Omega_c/\Omega)^{2N}}$$

Exercises

c



d $|G(\Omega_c)|^2 = \frac{1}{1+\epsilon^2}$.

We set Ω_c equal to $\Omega_p = 2\pi \cdot 50$, and $|G(\Omega_c)|^2$ equal to -1 dB.

e Determine ϵ by evaluating at $\Omega_p = 2\pi \cdot 50$:

$$|G(\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = 10^{-1/10} \qquad \Rightarrow \qquad \epsilon = \sqrt{10^{1/10} - 1} = 0.5088.$$

Exercises

Determine N by evaluating at $\Omega_s = 2\pi \cdot 40$:

$$|G(\Omega_s)|^2 = \frac{1}{1 + \epsilon^2 (\frac{2\pi \cdot 50}{2\pi \cdot 40})^{2N}} = 10^{-30/10} \qquad \Rightarrow \qquad (\frac{50}{40})^{2N} = \frac{10^{30/10} - 1}{\epsilon^2} = 3858$$

$$N = \frac{1}{2} \frac{\log(3858)}{\log(5/4)} = 18.5$$

The required filter order is $N = 19$.