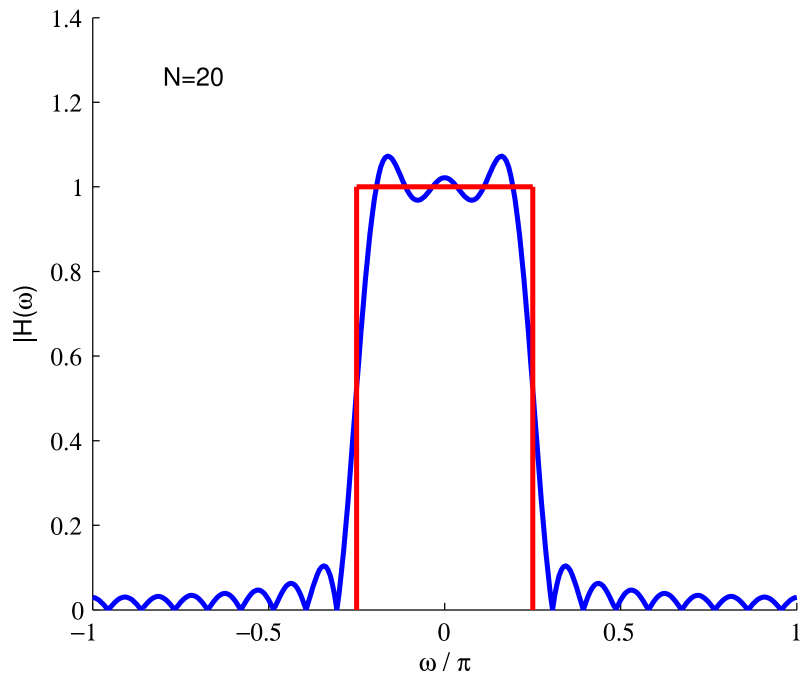
Ch.12 Digital filter design

How can I design a *digital* filter $H(z)$ that satisfies certain specifications?



Chaparro Ch. 12 t/m 12.5 (skip 12.4.4)

Digital filter design

Summary

Several techniques are available to design a digital filter:

- Specify a desired amplitude characteristic $|H(\omega)|$, and find the corresponding $h[n]$. For the phase, we require a linear phase characteristic. We will obtain an (anti-)symmetric FIR filter.
- First design an analog filter based on the given specifications (pass-band, damping in the stop-band). Then transform to the digital domain. This can be done by
  - sampling of the analog impulse response ("method of impulse invariance")
  - bilinear transform $s \to z$.

  This results in an IIR filter

IIR filters usually have a lower order for the same specifications, but they do not have linear phase (possibly resulting in pulse deformation in the pass-band).

Linear phase

Definition: a filter with frequency response

$$H(\omega) = A(\omega)e^{-j(\omega\alpha-\beta)}, \qquad -\pi \leq \omega \leq \pi$$

with $A(\omega)$ real, has *generalized linear phase*.

- The filter is similar to a delay for signals in the pass-band (where $A(\omega) \approx 1$), and does not distort these signals.
- Only FIR filters can have linear phase. Moreover, they must satisfy the symmetry property: $h[n] = \epsilon h[N-n]$, where $\epsilon = \pm 1$, and N is the filter order.
- The center of the impulse response is $N/2$. If N is even, this corresponds to a coefficient $h[N/2]$, else it doesn't. If $\epsilon = -1$ then $h[N/2] = 0$.

This results in four possibilities:

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- *Type I*: $\epsilon = 1$, N is even: $H(\omega) = e^{-j\omega N/2} \sum_k a_k \cos(\omega k)$

  $A(\omega) = A(-\omega)$, $\beta = 0$

- *Type II*: $\epsilon = 1$, N is odd: $H(\omega) = e^{-j\omega N/2} \sum_k a_k \cos(\omega(k - 1/2))$

  $H(\omega) = 0$ for $\omega = \pi$: cannot be a high-pass filter

  $A(\omega) = A(-\omega)$, $\beta = 0$

- *Type III*: $\epsilon = -1$, N is even: $H(\omega) = je^{-j\omega N/2} \sum_k a_k \sin(\omega k)$

  $H(\omega) = 0$ for $\omega = 0, \pi$: cannot be a low-pass nor a high-pass filter

  $A(\omega) = -A(-\omega)$, $\beta = \frac{\pi}{2}$

- *Type IV*: $\epsilon = -1$, N is odd: $H(\omega) = je^{-j\omega N/2} \sum_k a_k \sin(\omega(k - 1/2))$

  $H(\omega) = 0$ for $\omega = 0$: cannot be a low-pass filter

  $A(\omega) = -A(-\omega)$, $\beta = \frac{\pi}{2}$

The phase delay is $\alpha = N/2$, always equal to half the filter order.

Digital filter design

Design example

Design a low-pass filter with $\omega_p = 0.2\pi$, $\omega_s = 0.3\pi$, $\delta_p = \delta_s = 0.01$.

Approach (*"truncated impulse response design technique"*):

- Define the required amplitude response $A(\omega)$
- Select the type of filter: symmetric ($\epsilon = 1$) or anti-symmetric ($\epsilon = -1$), even or odd filter order
- Choose the filter order N; the corresponding phase characteristic is $e^{-j\omega N/2}$ of $je^{-j\omega N/2}$
- Apply an Inverse Discrete-Time Fourier Transform to obtain the impulse response
- Truncate at order N; hence we obtain an approximation

Digital filter design

Design example (2)

For our example: *the requested amplitude response* $A(\omega)$ *is:*

$$A(\omega) = \begin{cases} 1\,, & |\omega| < 0.25\pi \qquad = \frac{1}{2}(\omega_p + \omega_s) \\ 0\,, & \text{elders} \end{cases}$$

- *Select the phase characteristic*

  We select a filter with a symmetric impulse response, i.e., Type I or II. (The other types cannot give a low-pass filter.)

  The resulting phase function is $\phi(\omega) = \omega N/2$.

- *The desired transfer function is:*

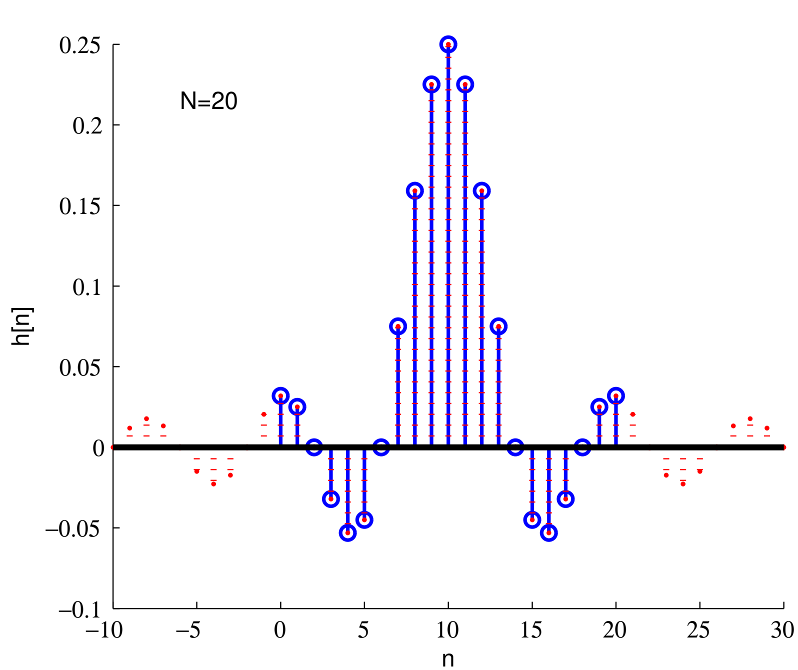
$$H_d(\omega) = \begin{cases} e^{-j\omega N/2}\,, & |\omega| < 0.25\pi \\ 0\,, & \text{elsewhere} \end{cases}$$

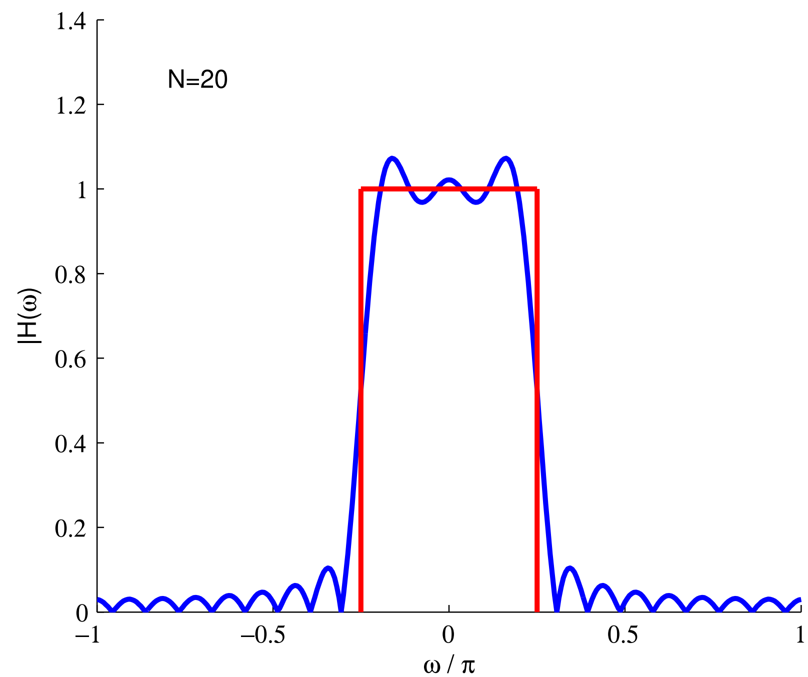
- The resulting impulse response is (IDTFT):

$$h_d[n] \;=\; \frac{1}{2\pi}\int_{-0.25\pi}^{0.25\pi} e^{j\omega(n-N/2)}\,d\omega \;=\; 0.25\,\text{sinc}[0.25(n-N/2)]$$

Design example (3)

We select for example a filter order $N = 20$. The impulse response will then be truncated to $n \in [0, 20]$. This will change the transfer function; we apply a DTFT to see what the resulting filter is in frequency domain.



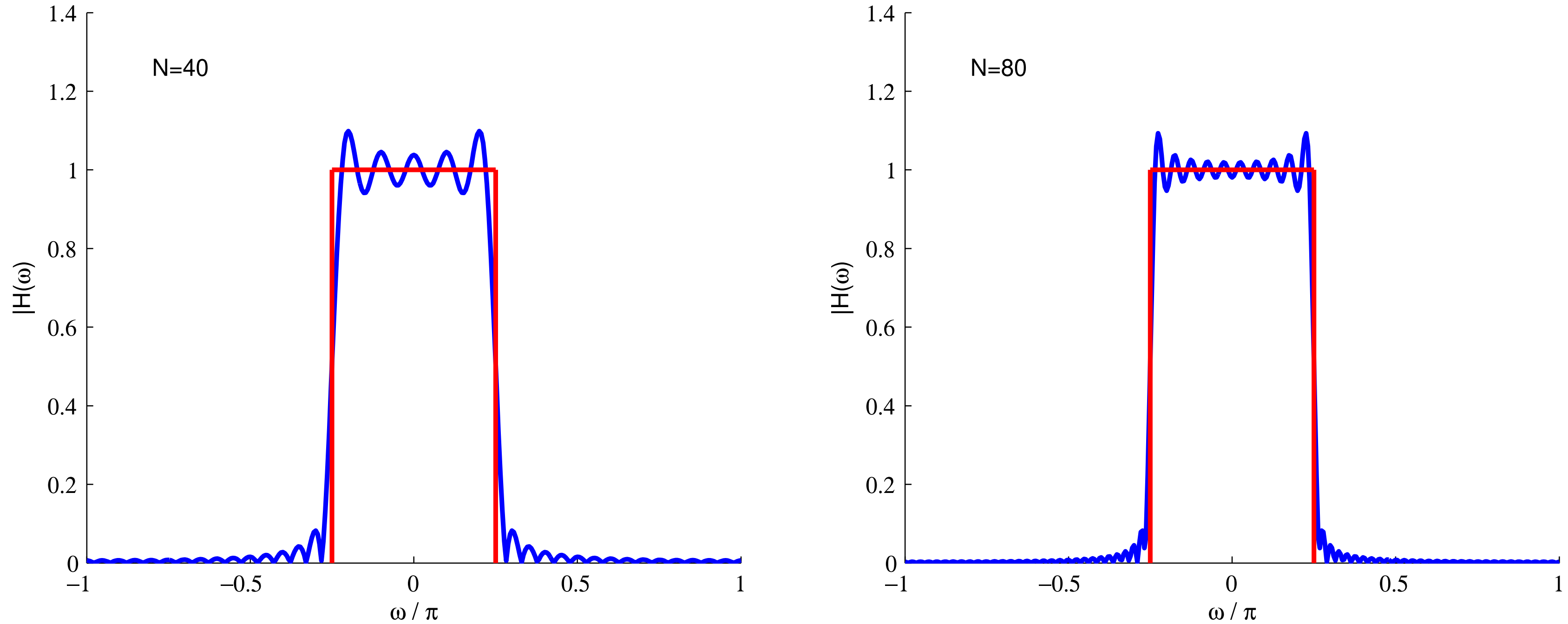


The pass-band and stop-band are OK, but the ripples are too large.

Digital filter design

Design example (4)

We can try to reduce the ripples by selecting a larger filter order (e.g., $N = 40$ of $N = 80$)



A larger N results in a smaller transition band, but the ripples have equal magnitude! This is the Gibb's effect.

Gibb's effect

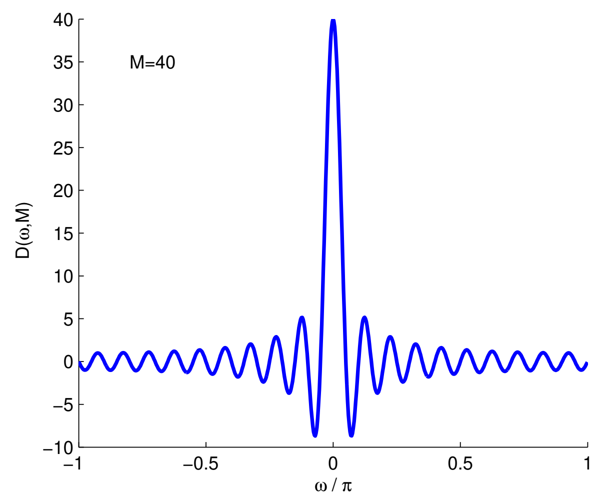
- Truncating an 'ideal' impulse response $h_d[n]$ to $h[n]$ corresponds to multiplication of $h[n]$ with a rectangular window $w_r[n]$:

$$h[n] = h_d[n]\, w_r[n]\,, \qquad w_r[n] = \begin{cases} 1\,, & n = 0, \cdots, N \\ 0\,, & \text{elsewhere} \end{cases}$$

- Multiplication in time domain corresponds to convolution in frequency domain: $H(\omega) = H_d(\omega) * W_r(\omega)$,

$$W_r(\omega) = \sum_{n=0}^{N} e^{-j\omega n} = \frac{1 - e^{-j\omega(N+1)}}{1 - e^{-j\omega}} = \frac{\sin(0.5\,\omega(N+1))}{\sin(0.5\,\omega)} e^{-j0.5\omega N}$$

The function $D(\omega, M) = \frac{\sin(0.5\omega M)}{\sin(0.5\omega)}$ is called the Dirichlet kernel. The magnitude of the first side lobe is approximately 9% of the peak value (-13 dB), independent of M. The main lobe has width $\approx 4\pi/M$ (first zero crossings).



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Other windows

We can try to reduce the Gibb's effect by selecting another window (not rectangular).

- Time domain: $h[n] = h_d[n]w[n]$.

  The window has to be symmetric: $w[n] = w[N-n]$, to keep the required symmetry of $h[n]$.

- Frequency domain: $H(\omega) = H_d(\omega) * W(\omega)$.

  Design criteria are:

  – The width of the main lobe of $W(\omega)$ should be as small as possible: this determines the width of the transition band.

    It usually is a multiple of $4\pi/M$, with $M = N + 1$. Thus, we can control this using the filter order.

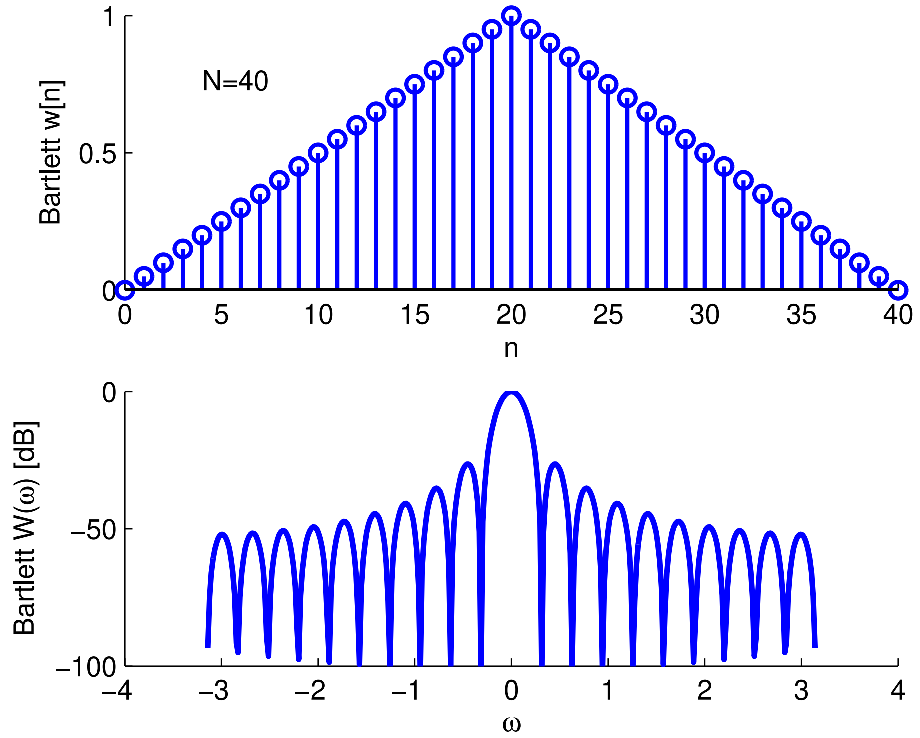
  – The amplitude of the first side lobe should be as small as possible.

Examples of windows

- **Bartlett window**: convolve the rectangular window with itself:

$$w_b = w_r * w_r \quad \Leftrightarrow \quad W_b(\omega) = W_r(\omega)^2$$

Width: $8\pi/M$, side lobe level $\delta_p, \delta_s = 0.05$ (-27 dB)



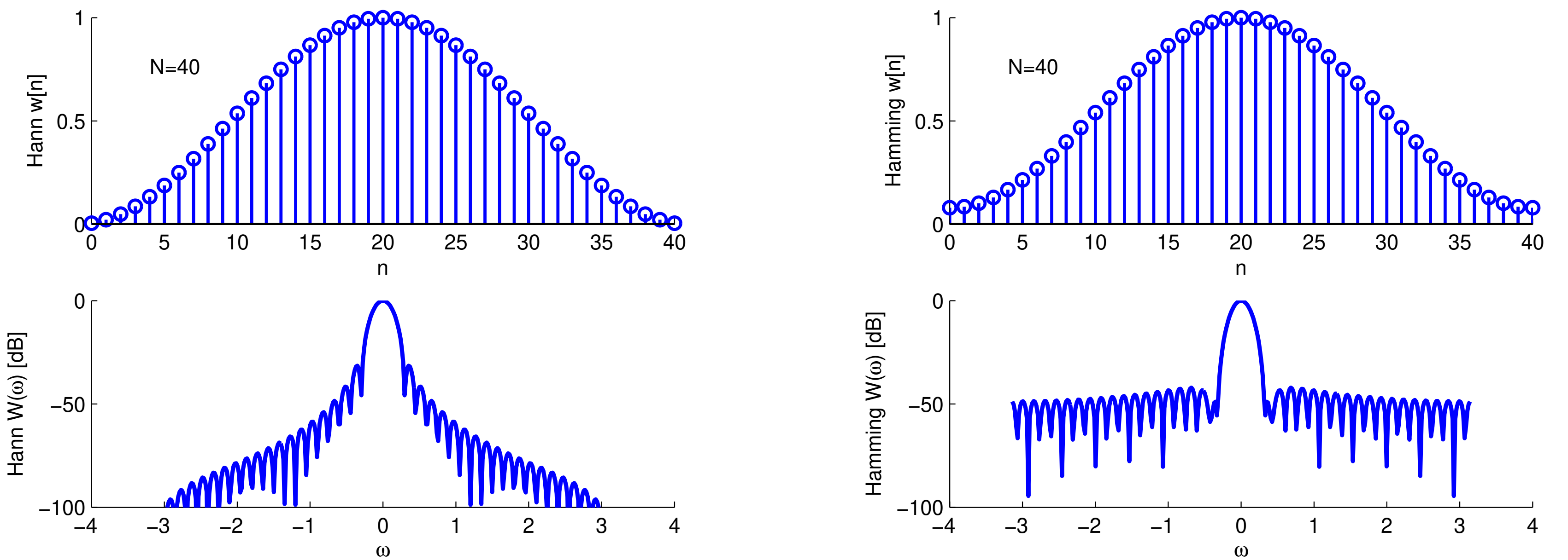
Examples of windows (cont'd)

- **Hann (Hanning?) window**: the weighted sum of three Dirichlet kernels, designed to reduce the side lobes. $w[n] = \frac{1}{2}(1 - \cos\frac{2\pi n}{N})w_r[n]$

  Width: $8\pi/M$, side lobe level $\delta_p, \delta_s = 0.0063$

- **Hamming window**: empirically better weighted sum of three Dirichlet kernels.

  Width: $8\pi/M$, side lobe level $\delta_p, \delta_s = 0.0022$. $w[n] = (0.54 - 0.46\cos\frac{2\pi n}{N})w_r[n]$



Digital filter design

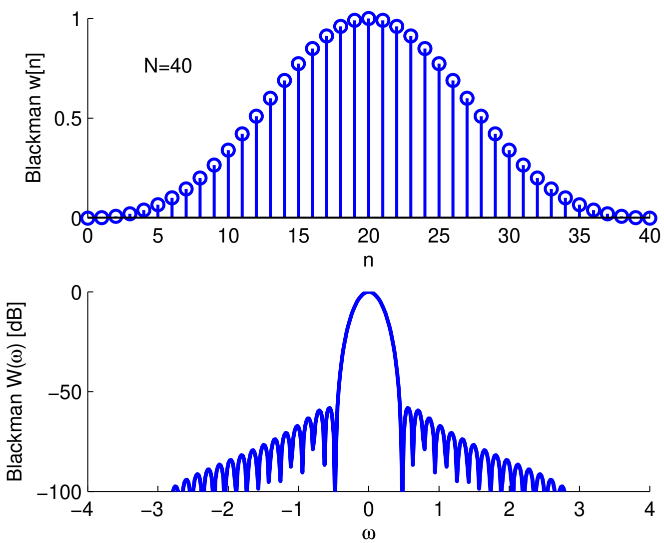
Examples of windows (cont'd)

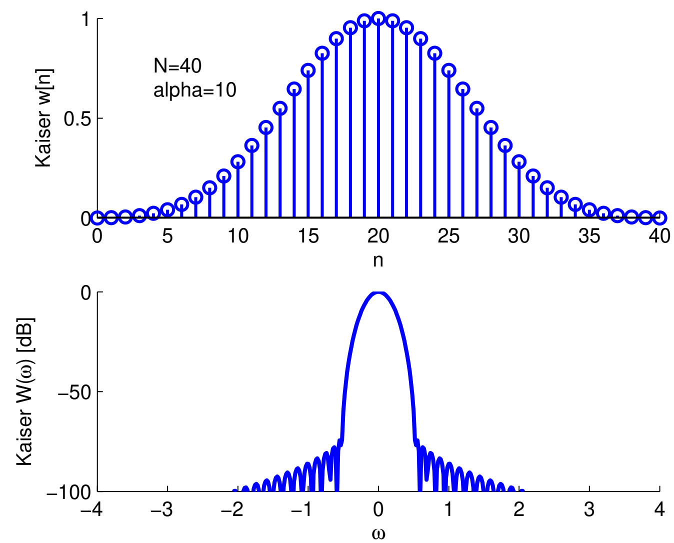
- **Blackman window**: weighted sum of five Dirichlet kernels

  Width: $12\pi/M$, side lobe level $\delta_p, \delta_s = 0.0002$

- **Kaiser window**: obtained by computer optimization (minimize the width of the peak, for a fixed energy of the side lobes).

  This design has a parameter α specifying the trade-off. E.g., $\alpha = 10$ gives width $12\pi/M$, side lobe level $\delta_p, \delta_s = 0.00001$





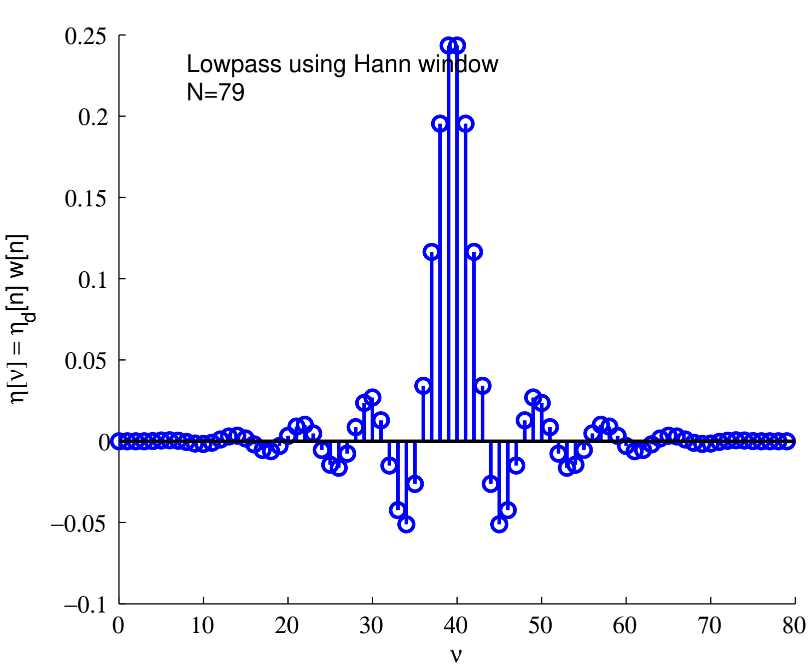
Digital filter design

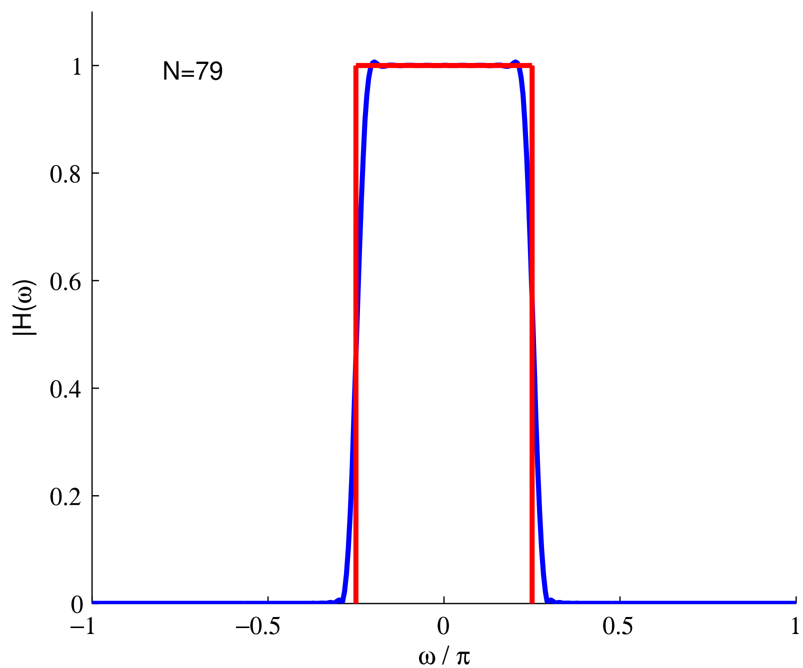
Returning to our filter design example:

- Design a low-pass filter with $\omega_p = 0.2\pi, \quad \omega_s = 0.3\pi, \quad \delta_p = \delta_s = 0.01.$

For $\delta_p = \delta_s = 0.01$, a Hann window suffices.

The transition band is $0.1\pi = 8\pi/M$, hence we can take $M = 80$, i.e., filter order $N = 79$.





Digital filter design

Design a digital differentiator

- *Analog:* $H_a(\Omega) = j\Omega$ *Digital:* $H(\omega) = j\frac{\omega}{T}$, $-\pi \leq \omega \leq \pi$,

with T: sample period.

Because of the 'j' (anti-symmetry) we need to take Type III or IV ($\epsilon = -1$).

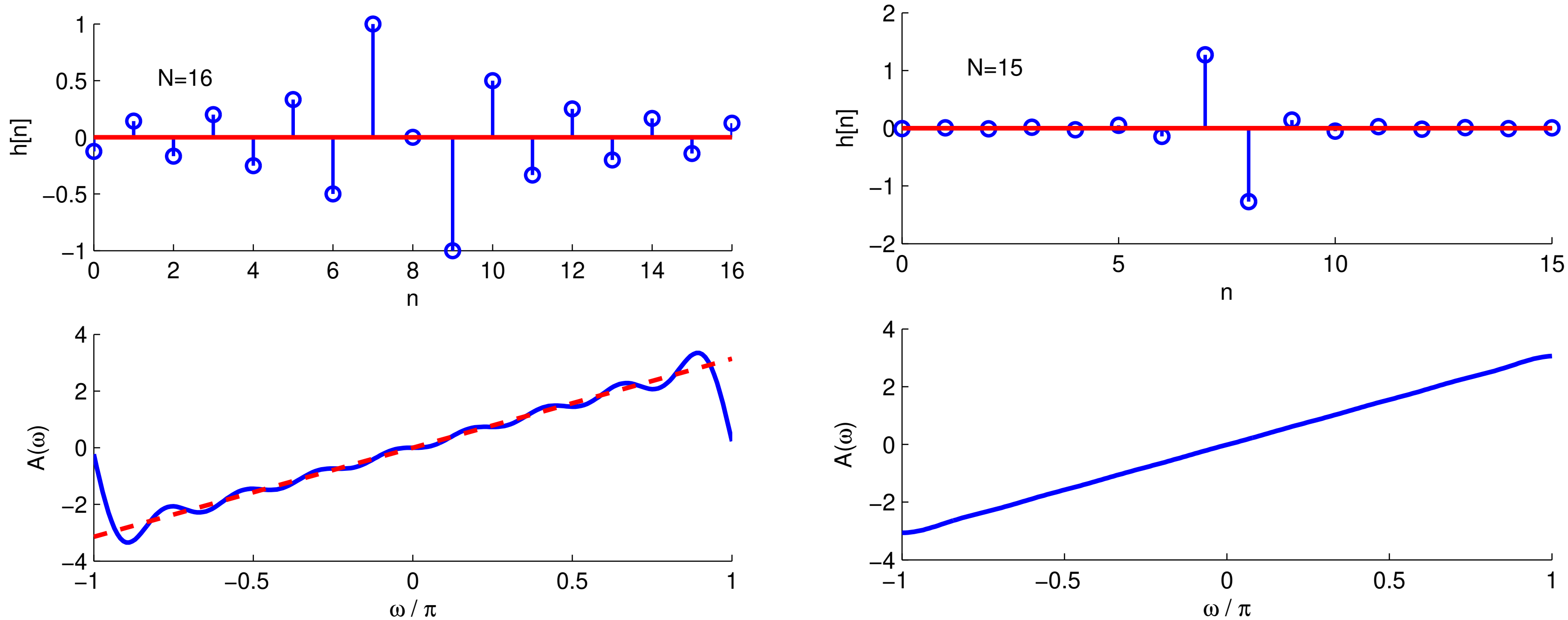
- *Resulting desired frequency response:*

$$H_d(\omega) = \frac{j\omega}{T} e^{-j(\omega N/2)} = \frac{\omega}{T} e^{j(0.5\pi - 0.5\omega N)}$$

- Corresponding impulse response:

$$h_d[n] = \frac{1}{2\pi T} \int_{-\pi}^{\pi} \omega e^{j(\omega n + 0.5\pi - 0.5\omega N)} d\omega = \begin{cases} \frac{(-1)^{(n-0.5N)}}{(n-0.5N)T}, & N \text{ even}, \; n \neq 0.5N \\ 0, & N \text{ even}, \; n = 0.5N \\ \frac{(-1)^{(n-0.5N+0.5)}}{\pi(n-0.5N)^2 T}, & N \text{ odd}. \end{cases}$$

Design a digital differentiator (2)



- An odd filter order N (type IV) results in a much faster decay of $h[n]$, because of the square in the denominator.

  This is because for even N, the amplitude response is anti-symmetric and periodic in 2π, resulting in $A(\pm\pi) = 0$ (not desired for a high-pass characteristic). Hence, Type III is not suitable.

Digital filter design

In practice, we use a computer program for FIR filter design

Often used: Parks/McClellan technique, also known as the Remez exchange algoritm

- Specify pass-band and stop-band, e.g., $F = [0, 0.4, 0.6, 1]$ specifies a pass-band from 0 until $\omega_p = 0.4\pi$, a stop-band from $\omega_s = 0.6\pi$ until π.
- Specify the desired reponse at these critical frequencies (F), e.g., $H_d = [1, 1, 0, 0]$.
- Specify the ripple, as a weight vector $W = [1/\delta_p, 1/\delta_s]$.
- Select the filter order N (using rules of thumb; trial and error)

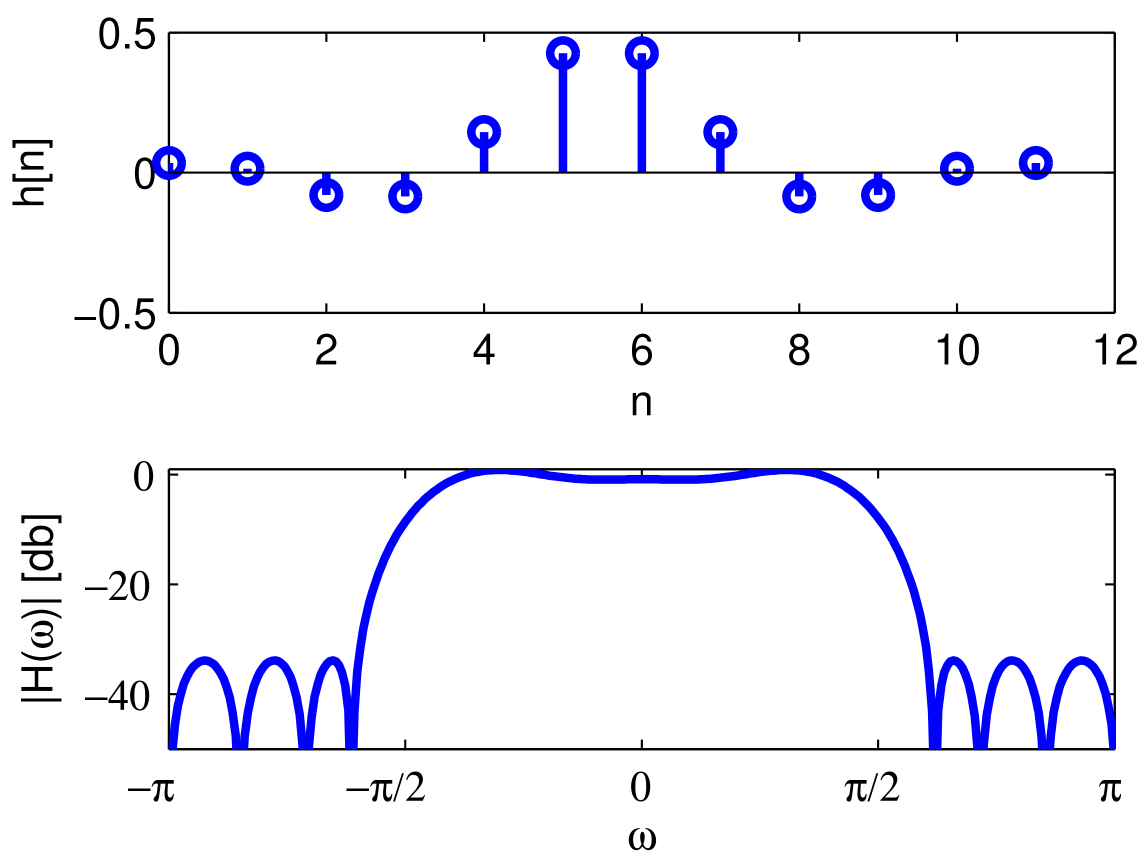
The algoritm searches $h[n]$ such that $\max_{\omega} \|W(\omega)\,(H_d(\omega) - A(\omega))\,\|$ is as small as possible ("minimax" optimization).

Digital filter design

Matlab: `h = remez(N,F,H_d,W)`

(function is now called `firpm`)

Result ($N = 11, \delta_p = 0.05, \delta_s = 0.01$):



Digital filter design

Alternative (IIR filters): method of impulse invariance

- First design a suitable *analog* filter. E.g., a Butterworth,

$$|H_a(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} \quad \Rightarrow \quad H_a(s) = \frac{(\Omega_c)^N}{(s-s_0)(s-s_1)\cdots(s-s_{N-1})}$$

  (Here, $s = j\Omega$, and Ω_c is the cut-off frequency, the poles s_i are on a circle with radius Ω_c)

- Sample the corresponding impulse response $h_a(t)$ with period T_s:

$$h[n] = h_a(nT_s)$$

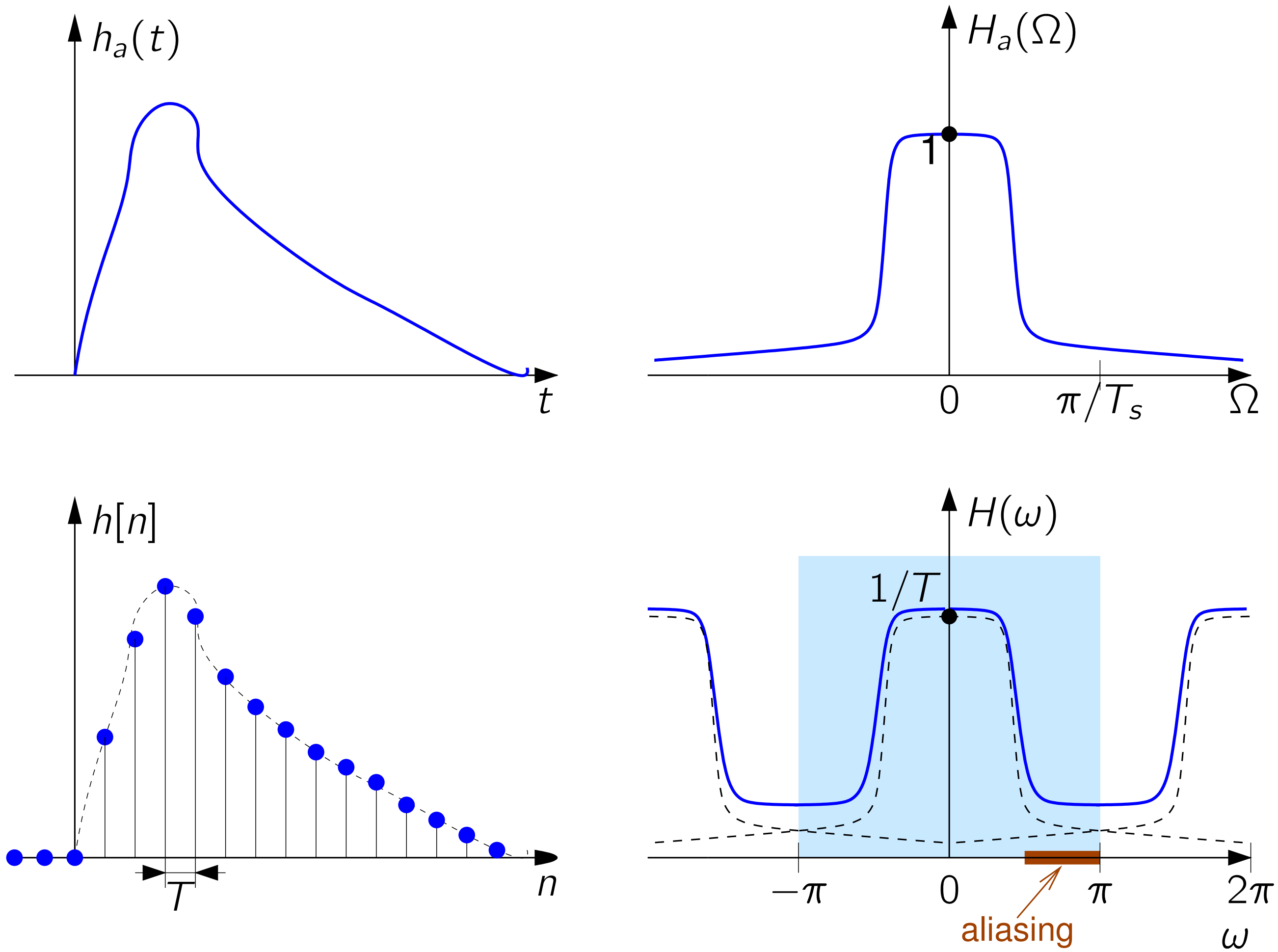
During sampling, *aliasing* can occur:

$$H(\omega) = \frac{1}{T_s}\sum_k H_a(\Omega - \frac{2\pi k}{T_s}), \qquad \Omega = \frac{\omega}{T_s}$$

The frequency response in digital domain is periodic. Without aliasing,

$$H(\omega) = \frac{1}{T_s} H_a(\frac{\omega}{T_s}), \quad |\omega| < \pi$$

Method of impulse invariance



This technique is not suitable for high-pass characteristics because of aliasing.

Digital filter design

Method of impulse invariance

If we know $H_a(s)$, we do not need to first compute $h_a(t)$.

- Determine the poles of $H_a(s)$:

$$H_a(s) = \sum_k \frac{A_k}{s - s_k}$$

- The corresponding analog impulse response is

$$h_a(t) = \sum_k A_k e^{s_k t} u(t)$$

  with $u(t)$ a unit step function.

- The sampled version is

$$h[n] = \sum_k A_k e^{s_k n T_s} u[n] = \sum_k A_k (e^{s_k T_s})^n u[n]$$

- The corresponding z-transform is

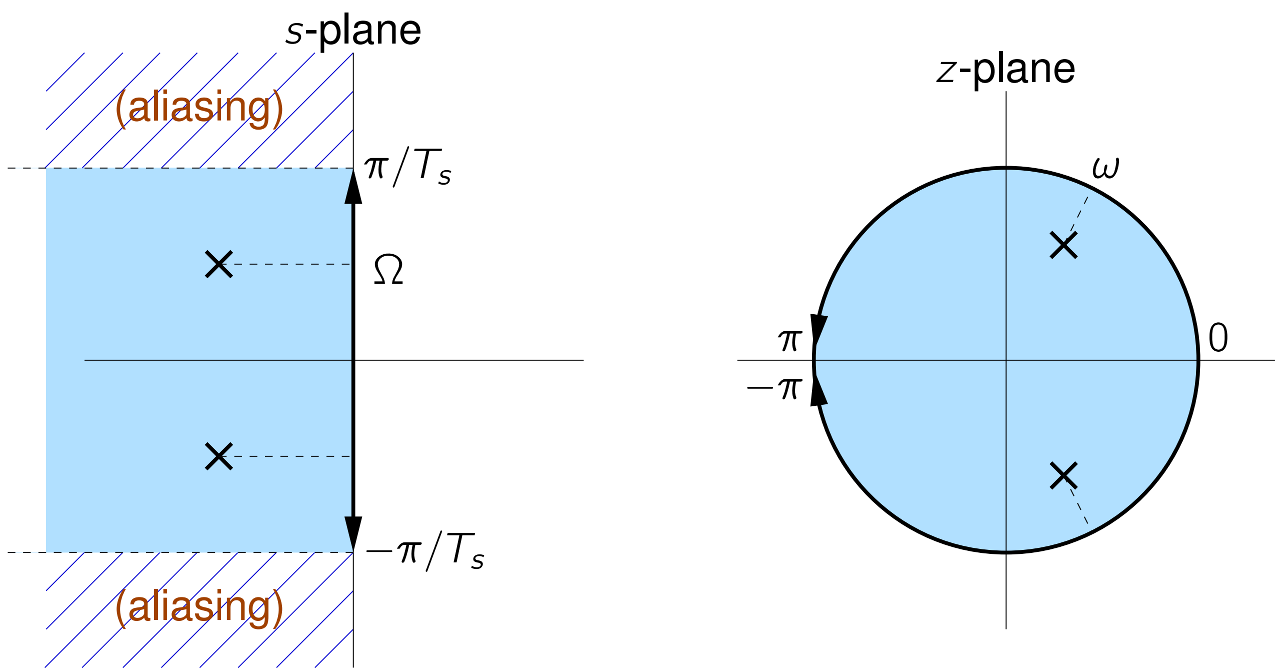
$$H(z) = \sum \frac{A_k}{1 - p_k z^{-1}}, \qquad p_k = e^{s_k T_s}$$

Method of impulse invariance

Hence:

- The filter order is constant,
- A 'stable pole' s_k in the left half plane ($\text{Re}(s_k) < 0$) is transformed to a pole $p_k = e^{s_k T_s}$ within the unit circle ($|p_k| < 1$).

  Causality and stability are preserved.



Second alternative: bilinear transform

Given an analog filter $H_a(s)$. We can transform this into a digital filter by the mapping

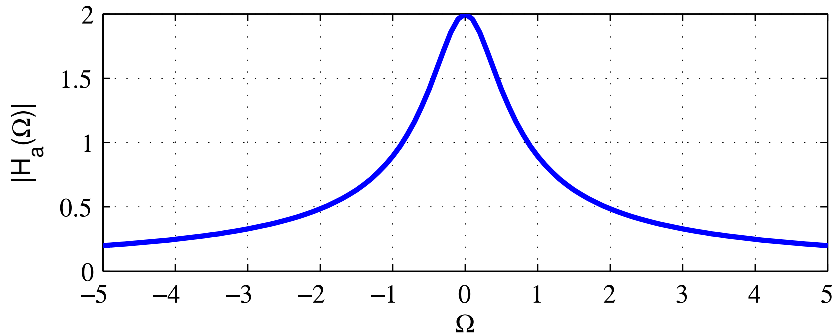
$$H(z) = H_a(s)\,, \qquad \text{with} \quad s := \frac{1 - z^{-1}}{1 + z^{-1}}$$

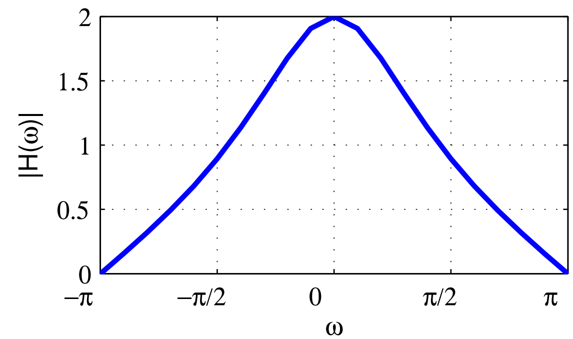
- Example (with $b > 0$)

$$H_a(s) = \frac{1}{s+b} \qquad \rightarrow$$

$$H(z) = \frac{1}{\frac{1-z^{-1}}{1+z^{-1}} + b} = \frac{1}{1+b} \cdot \frac{1+z^{-1}}{1 - \frac{1-b}{1+b} z^{-1}}$$

The pole at $s = -b$ is mapped to a pole at $z = \rho = \dfrac{1-b}{1+b}$.





Digital filter design

Derivation: bilinear transform

Simple example: consider an integrator $H_a(s) = \frac{Y(s)}{X(s)} = \frac{1}{s}$, or

$$y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)d\tau$$

Approximate the integral using a trapezium rule:

$$y(nT) \approx y((n-1)T) + \frac{T}{2}[x(nT) + x((n-1)T)]$$

The corresponding z-transform gives

$$Y(z) = z^{-1}Y(z) + \frac{T}{2}[X(z) + z^{-1}X(z)]$$

with transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

The same result is obtained by substituting in $H_a(s) = 1/s$ the s by

$$s \to \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

This is called the *bilinear transform*.

Properties of the bilinear transform

$$s = \frac{1 - z^{-1}}{1 + z^{-1}} \qquad \Leftrightarrow \qquad z = \frac{1 + s}{1 - s}$$

- If $s = \sigma + j\Omega$ with $\sigma < 0$, then $|z| < 1$
- If $H_a(s)$ has a pole at $s = s_k$, then $H(z)$ has a pole at $p_k = \dfrac{1 + s_k}{1 - s_k}$

  If $H_a(s)$ is causally stable, then also $H(z)$. The filter order remains the same.
- The imaginary axis $s = j\Omega$ is mapped one-to-one to the unit circle $|z| = 1$:

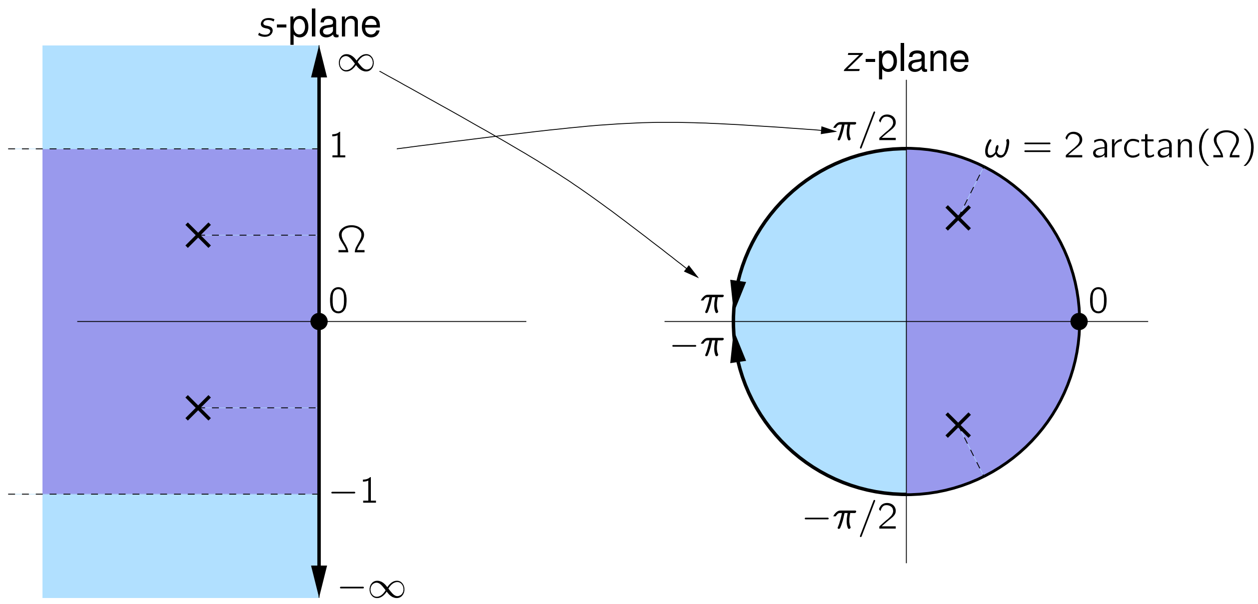
$$s = j\Omega \qquad \Leftrightarrow \qquad z = \frac{1 + j\Omega}{1 - j\Omega} \quad = \frac{A}{A^*} \qquad \Rightarrow \qquad |z| = 1$$

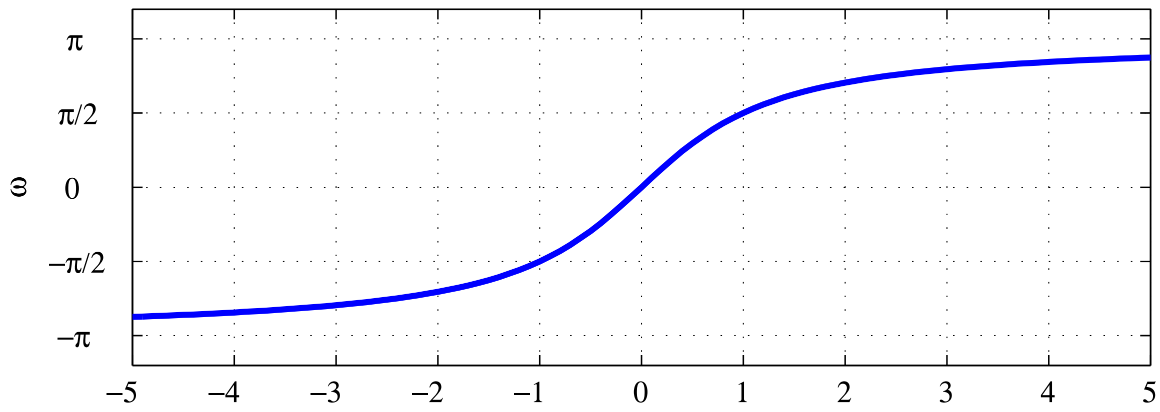
- With $s = j\Omega$ and $z = e^{j\omega}$ we find

$$\omega = 2\arctan(\Omega) \qquad \Leftrightarrow \qquad \Omega = \tan(\frac{\omega}{2})$$

  Thus, the Ω-axis is mapped non-linearly to the unit circle: high frequencies ($\Omega \to \infty$) are compressed towards $\omega \to \pi$. No aliasing but a deformation. Until $\Omega = 1$, the mapping is approximately linear.

Properties of the bilinear transform





Digital filter design via analog filter design

Transformation from analog to digital filter

After designing an analog filter, we can transform this to a digital filter. We have seen these transforms:

- impulse invariance (not suitable for high-pass or band-stop)
- bilinear transform

Alternative: first design an analog low-pass filter, transform to a digital filter, apply a frequency transformation in digital domain.

- (We do not discuss these transforms)
- Also suitable for high-pass or band-stop
- Not equivalent, except for bilinear transform

Design specifications for a digital filter first have to be translated to specs for an analog filter. After the design we return to the digital domain via the selected transform.

Digital filter design via analoog filter design

Example: design using digital specifications

- Design a first-order digital low-pass filter with 3-dB band-width at

  $\omega_c = 0.2\pi$.

- Solution:
  - Bilinear transform of ω_c to the analog frequency domain:

    $\Omega_c = \tan(\omega_c/2) = 0.325$

  - Design a first-order Butterworth filter:

$$|H_a(s)|^2 = \frac{1}{1+(\Omega/\Omega_c)^2} = H_a(s)H_a(-s)\Big|_{s=j\Omega} \quad \Rightarrow \quad H_a(s) = \frac{\Omega_c}{s+\Omega_c}$$

  - Bilinear transform of $H_a(s)$ back to $H(z)$

$$s = \frac{1-z^{-1}}{1+z^{-1}}, \qquad H(z) = \frac{\Omega_c}{\frac{1-z^{-1}}{1+z^{-1}} + \Omega_c} = \frac{\Omega_c(1+z^{-1})}{1+\Omega_c - z^{-1}(1-\Omega_c)} = \frac{0.245(1+z^{-1})}{1-0.509\,z^{-1}}$$

  - Check: $|H(\omega=0)| = 1, \quad |H(\omega=0.2\pi)|^2 = 1/2.$