Exercises Ch.10 z-transform

Exercise 1 – exercises taken from Chaparro (first edition)

Mapping of s-plane into the z-plane

The poles of the Laplace transform $X(s)$ of an analog signal $x(t)$ are

$$p_{1,2} = -1 \pm j1$$

$$p_3 = 0$$

$$p_{4,5} = \pm j1$$

There are no zeros. If we use the transformation $z = e^{sT_s}$ with $T_s = 1$:

(a) Determine where the given poles are mapped into the z-plane.

(b) How would you determine if these poles are mapped inside, on, or outside the unit circle in the z-plane? Explain.

(c) Carefully plot the poles and the zeros of the analog and the discrete-time signals in the Laplace and the z-planes.

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answer

Pr 9.1 (a)(c) For $T_s = 1$ the transformation from the s-plane to the z-plane $z = e^s$ is such that for $s = \sigma + j\Omega$

$$z = e^{\sigma} e^{j\Omega}$$

For $s = -1 \pm j1$, the poles of the analog system, the corresponding singularities in the z-plane are given by

$$z = e^{-1} e^{\pm j\,1}$$

which are inside the unit disk as $e^{-1} < 1$ with a radian frequency of ± 1
The pole $s = 0$ is mapped into $z = e^0 = 1$, and the poles $s = \pm j1$ are mapped into $z = 1e^{\pm j1}$ with unit magnitude and radian frequencies ± 1.
(b) By expressing $z = re^{j\omega} = e^{\sigma} e^{j\Omega}$ for $T_s = 1$, the radius is given by $r = e^{\sigma}$ so that if $\sigma < 0$ the singularities are inside the unit circle, if $\sigma = 0$ they are on the unit circle, and if $\sigma > 0$ they are outside the unit circle.

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Exercise 2

Mapping of z-plane into the s-plane

Consider the inverse relation given by $z = e^{sT_s}$—that is, how to map the z-plane into the s-plane.

(a) Find an expression for s in terms of z from the relation $z = e^{sT_s}$.

(b) Consider the mapping of the unit circle (i.e., $z = 1e^{j\omega}$, $-\pi \leq \omega < \pi$). Obtain the segment in the s-plane resulting from the mapping.

(c) Consider the mapping of the inside and the outside of the unit circle. Determine the regions in the s-plane resulting from the mappings.

(d) From the above results, indicate the region in the s-plane to which the whole z-plane is mapped into. Since $\omega = \omega + 2\pi$, is this mapping unique? Explain.

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answer

Pr 9.2 (a) Solving for s in the given relation we obtain

$$s = \log(z)/T_s.$$

(b) Points on the unit circle that in the z-plane are represented by $z = 1e^{j\omega}$ (unit radius and frequency $-\pi \leq \omega < \pi$) will be mapped into

$$s = \frac{\log(1e^{j\omega})}{T_s} = \frac{0 + j\omega}{T_s} = j\frac{\omega}{T_s} = j\Omega$$

or points in the $j\Omega$ axis of the s-plane. For instance $z = 1 = 1e^{j0}$ is mapped into $s = j0$, the origin of the s-plane; $z = 1e^{\pm j\pi/2}$ is mapped into $s = \pm j\pi/(2T_s)$ and $z = -1 = 1e^{\pm j\pi}$ maps into $s = \pm j\pi/T_s$. Thus, the unit disk is mapped into a line in the $j\Omega$–axis from $-\pi/T_s$ to π/T_s.
(c) In general, $z = re^{j\omega}$ is mapped into

$$s \;=\; \frac{\log[re^{j\omega}]}{T_s} = \underbrace{\frac{\log[r]}{T_s}}_{\sigma} + j\underbrace{\frac{\omega}{T_s}}_{\Omega}$$

where we used $r = e^{\sigma T_s}$ and $\omega = \Omega T_s$ giving us the correct expression for $s = \sigma + j\Omega$. The outside of the unit circle, i.e., $r > 1$ maps into the right-hand strip defined by $\sigma > 0$ and $-\pi/T_s \leq \Omega < \pi/T_s$, the inside of the unit circle, i.e, $r \leq 1$ maps into the left-hand strip defined by $\sigma < 0$ and $-\pi/T_s \leq \Omega < \pi/T_s$.

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answer

(d) The above shows that the z-plane is mapped into the strip between $-\pi/T_s$ and π/T_s for all σ in the s-plane. Adding $2\pi k$, multiples of 2π, the discrete frequencies remain the same, i.e., $\omega = \omega + 2\pi k$ for $k = 0, \pm 1, \pm 2, \cdots$. The mapping of the z-plane with these frequencies gives the same values of σ as before, but the analog frequencies will be the mapping of $(2k-1)\pi \leq \omega < (2k+1)\pi$:

$$\frac{(2k-1)\pi}{T_s} \leq \Omega < \frac{(2k+1)\pi}{T_s}$$

corresponding to strips of width $2\pi/T_s$ above (for $k \geq 1$) and below (for $k \leq -1$) the one we considered before. Thus, the z-plane is mapped separately into strips of the same width in the s-plane.

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Exercise 3

Z-transform and ROCs

Consider the noncausal sequence

$$s[n] = s_1[n] + s_2[n]$$

where $s_1[n] = u[n]$ is causal and $s_2[n] = -u[-n]$ is anti-causal. This signal is the signum, or sign function, that extracts the sign of a real-valued signal—that is,

$$s[n] = \text{sgn}(x[n]) = \begin{cases} -1 & x[n] < 0 \\ 0 & x[n] = 0 \\ 1 & x[n] > 0 \end{cases}$$

(a) Find the Z-transforms of $s_1[n]$ and $s_2[n]$, indicating the corresponding ROC.

(b) Determine the Z-transform $S(z)$.

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answer

<u>Pr 9.3</u> (a) The Z-transform of $s_1[n]$ is

$$S_1(z) = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$

provided that $|z^{-1}| < 1$, thus $|z| > 1$ is the region of convergence for $S_1(z)$. The Z-transform of $s_2[n]$ is given by

$$S_2(z) = -\sum_{n=-\infty}^{0} z^{-n} = -\sum_{m=0}^{\infty} z^{m} = \frac{-1}{1-z}$$

where the last sum converges for $|z| < 1$.

(b) The condition for

$$S(z) = S_1(z) + S_2(z) = \frac{1}{1-z^{-1}} - \frac{1}{1-z} = \frac{1+z^{-1}}{1-z^{-1}}$$

to converge is that $|z| > 1$ and that $|z| < 1$ simultaneously, which is not possible. Since there is no region of convergence for $S(z)$, the Z-transform of $s[n]$ does not exist.

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Exercise 4

Z-transform and ROC

Given the anti-causal signal

$$x[n] = -\alpha^n u[-n]$$

(a) Determine the Z-transform $X(z)$, and carefully plot the ROC when $\alpha = 0.5$ and $\alpha = 2$. For which of the two values of α does $X(e^{j\omega})$ exist?

(b) Find the signal that corresponds to the derivative $dX(z)/dz$. Express it in terms of α.

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answer

Pr 9.4 (a) The signal $x[n]$ can be written as

$$\begin{aligned} x[n] &= -\delta[n] - \alpha^{-1}\delta[n+1] - \alpha^{-2}\delta[n+2] - \cdots \\ &= -\delta[n] - \frac{1}{\alpha}\delta[n+1] - \frac{1}{\alpha^2}\delta[n+2] - \cdots \end{aligned}$$

So that its Z-transform is given by

$$\begin{aligned} X(z) &= -1 - \frac{z}{\alpha} - \frac{z^2}{\alpha^2} - \cdots \\ &= -\sum_{n=0}^{\infty}\left(\frac{z}{\alpha}\right)^n = -\frac{1}{1-z/\alpha} = \frac{\alpha z^{-1}}{1-\alpha z^{-1}} \qquad |z| < |\alpha| \end{aligned}$$

If $\alpha = 0.5$, the ROC is the interior of a circle of radius 0.5, which does not include the unit circle. The ROC in this case indicates that the signal is non-causal. If $\alpha = 2$, the ROC is the interior of a circle of radius 2, including the unit circle, and indicating the signal is non-causal. In this case $X(e^{j\omega})$ is defined, but it is not when $\alpha = 0.5$.

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(b) Computing the derivative of $X(z)$ with respect to z gives

$$\begin{aligned}\frac{d\,X(z)}{dz} &= -\frac{1}{\alpha}-\frac{2z}{\alpha^2}-\frac{3z^2}{\alpha^3}\cdots=-\sum_{n=1}^{\infty}n\frac{z^{n-1}}{\alpha^n}\\ &= \sum_{m=-\infty}^{-1}m\alpha^m z^{-(m+1)}\end{aligned}$$

by letting $m=-n$ in the last sum. Letting now $k=m+1$ in the final sum we have

$$\frac{d\,X(z)}{dz}=\sum_{k=-\infty}^{0}(k-1)\alpha^{(k-1)}z^{-k}$$

We thus have the pair

$$\frac{d\,X(z)}{dz}\quad\Leftrightarrow\quad(n-1)\alpha^{(n-1)}u[-n].$$

Writing $X(z)$ in positive powers of z, i.e., $X(z)=\alpha/(z-\alpha)$ its derivative with respect to z is

$$\frac{d\,X(z)}{dz}=\frac{-\alpha}{(z-\alpha)^2}$$

so that we have

$$\frac{-\alpha}{(z-\alpha)^2}=\frac{-\alpha z^{-2}}{(1-\alpha z^{-1})^2}\quad\Leftrightarrow\quad(n-1)\alpha^{(n-1)}u[-n].$$

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Exercise 5

Significance of ROC

Consider a causal signal $x_1[n] = u[n]$ and an anti-causal signal $x_2[n] = -u[-n-1]$.

(a) Find the Z-transforms $X_1(z)$ and $X_2(z)$ and carefully plot their ROCs. If the ROCs are not included with the Z-transforms, would you be able to tell which is the correct inverse? Explain.

(b) Determine if it is possible to find the Z-transform of $x_1[n] + x_2[n]$.

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answer

Pr 9.5 (a) The Z-transform of $x_1[n]$ is

$$X_1(z) = \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

Using

$$u[-n-1] = \begin{cases} 1 & n \leq -1 \\ 0 & n \geq 0 \end{cases}$$

the Z-transform of $x_2[n]$ is

$$\begin{aligned} X_2(z) &= -\sum_{n=-\infty}^{-1} z^{-n} \\ &= -\sum_{m=0}^{\infty} z^m + 1 \\ &= \frac{-1}{1-z} + 1 \\ &= \frac{1}{1 - z^{-1}} \qquad |z| < 1 \end{aligned}$$

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answer

where we changed the variable in the sum, $m = -n$, and started the sum at zero instead of 1, and added 1 so it would not change. For $X_2(z)$ to exist, it is required that $|z| < 1$.

If we ignore the regions of convergence, we would then have that $1/(1 - z^{-1})$ would be the Z-transform of two completely different signals, $x_1[n]$ and $x_2[n]$, rendering the transformation as useless.

(b) For the Z-transform of $x_1[n] + x_2[n]$ to exist, requires that its ROC be the intersection of those of $X_1(z)$ and $X_2(z)$. The intersection of $|z| < 1$ and $|z| > 1$ is empty, so the Z-transform does not exist.

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Exercise 6

Laplace and Z-transforms of sampled signals

An analog pulse $x(t) = u(t) - u(t-1)$ is sampled using a sampling period $T_s = 0.1$.

(a) Obtain the discrete-time signal $x(nT_s) = x(t)|_{t=nT_s}$ and plot it as a function of nT_s.

(b) If the sampled signal is represented as an analog signal as

$$x_s(t) = \sum_{n=0}^{N-1} x(nT_s)\delta(t - nTs)$$

determine the value of N in the above equation.

(c) Compute the Laplace transform of the sampled signal (i.e., $X_s(s) = \mathcal{L}[x_s(t)]$).

(d) Determine the Z-transform of $x(nT_s)$, or $X(z)$.

(e) Indicate how to transform $X_s(s)$ into $X(z)$

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Pr 9.7 (a) If $T_s = 0.1$ the discrete-time signal is

$$x(0.1n) = [u(t) - u(t-1)]\,|_{t=0.1n} = \begin{cases} 1 & 0 \le 0.1n \le 1 \quad \text{or} \quad 0 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

(b) Expressing $x[n]$ as indicated, then $N = 11$.

(c) The Laplace transform of the sampled signal is

$$\begin{aligned} X_s(s) &= \sum_{n=0}^{10} \mathcal{L}[\delta(t - nT_s)) \\ &= \sum_{n=0}^{10} e^{-0.1ns} \\ &= \frac{1 - e^{-1.1s}}{1 - e^{-0.1s}} \end{aligned}$$

(d) The z-transform of the discrete-time signal is

$$X(z) = \sum_{n=0}^{10} z^{-n} = \frac{1 - z^{-11}}{1 - z^{-1}}$$

(e) To transform $X_s(s)$ into $X(z)$ we let $z = e^{0.1s}$.

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Exercise 7

Computation of Z-transform

A causal exponential $x(t) = 2e^{-2t}u(t)$ is sampled using a sampling period $T_s = 1$. The corresponding discrete-time signal is $x[n] = 2e^{-2n}u[n]$.

(a) Express the discrete-time signal as $x[n] = 2\alpha^n u[n]$ and give the value of α.

(b) Find the Z-transform $X(z)$ of $x[n]$ and plot its poles and zeros in the z-plane.

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answer

Pr 9.9 (a) The discrete-time signal $x[n] = 2e^{-2n}u[n]$ can be equally written

$$x[n] = 2(e^{-2})^n = 2\alpha^n$$

or $\alpha = e^{-2} < 1$.

(b) The z-transform of $x[n]$ is

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} 2\alpha^n z^{-n} \\ &= \sum_{n=0}^{\infty} 2(\alpha z^{-1})^n \\ &= \frac{2}{1-\alpha z^{-1}} \qquad |\alpha z^{-1}| < 1 \quad \text{or} \quad |z| > |\alpha| \end{aligned}$$

To find poles and zeros let

$$X(z) = \frac{2z}{z-\alpha}$$

with zero $z = 0$ and pole $z = \alpha$.

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Exercise 8

Computation of Z-transform

Consider the signal $x[n] = 0.5(1 + [-1]^n)u[n]$.

(a) Plot $x[n]$ and use the definition of the Z-transform to obtain its Z-transform, $X(z)$.

(b) Use the linearity property and the Z-transforms of $u[n]$ and $[-1]^n u[n]$ to find the Z-transform $X(z) = \mathcal{Z}[x[n]]$.

(c) Determine and plot the poles and the zeros of $X(z)$.

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answer

Pr 9.10 (a) The given signal can also be written

$$x[n] = \begin{cases} 1 & n \geq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases}$$

(b) Using the above expression for $x[n]$, we have

$$\begin{aligned} X(z) &= \sum_{n=0,\ \text{even}}^{\infty} 1\, z^{-n} \\ &= \sum_{m=0}^{\infty} 1\, z^{-2m} = \frac{1}{1 - z^{-2}} \qquad |z| > 1 \end{aligned}$$

where we let $n = 2m$ to find the final expression.

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answer

(c) The z-transform of $x[n]$ is also obtained by using its linearity

$$\begin{aligned} X(z) &= 0.5\mathcal{Z}[u[n]] + 0.5\mathcal{Z}[(-1)^n u[n]] \\ &= \frac{1}{2(1-z^{-1})} + 0.5\sum_{n=0}^{\infty}(-z^{-1})^n \\ &= \frac{1}{2(1-z^{-1})} + \frac{1}{2(1+z^{-1})} \\ &= \frac{1}{1-z^{-2}} \qquad |z^{-1}| < 1 \quad \text{or} \quad |z| > 1 \end{aligned}$$

(c) To find the poles and zeros let

$$X(z) = \frac{z^2}{z^2 - 1}$$

with poles $z = \pm 1$, and zeros $z = 0$, double.

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Exercise 9

Solution of difference equations with Z-transform
Consider a system represented by the first-order difference equation

$$y[n] = x[n] - 0.5y[n-1]$$

where $y[n]$ is the output and $x[n]$ is the input.

(a) Find the Z-transform $Y(z)$ in terms of $X(z)$ and the initial condition $y[-1]$.

(b) Find an input $x[n] \neq 0$ and an initial condition $y[-1] \neq 0$ so that the output is $y[n] = 0$ for $n \geq 0$. Verify you get this result by solving the difference equation recursively.

(c) For zero initial conditions, find the input $x[n]$ so that $y[n] = \delta[n] + 0.5\delta[n-1]$.

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Pr 9.11 (a) The Z-transform of the difference equation

$$y[n] = x[n] - 0.5y[n-1] \qquad n \geq 0$$

with initial condition $y[-1]$ is

$$Y(z) = X(z) - 0.5(z^{-1}Y(z) + y[-1])$$

so that

$$Y(z) = \frac{X(z)}{1+0.5z^{-1}} - \frac{0.5y[-1]}{1+0.5z^{-1}}$$

(b) If $X(z) = 1$, $y[-1] = 2$ then $Y(z) = 0$ and therefore $y[n] = 0$ for $n \geq 0$ but $y[-1] = 2$. If $X(z) = 1$ or $x[n] = \delta[n]$ and $y[-1] = 2$ the difference equation is

$$y[n] = \delta[n] - 0.5y[n-1] \qquad n \geq 0$$

and can be solved recursively

$$\begin{aligned} y[0] &= 1 - 0.5 \times 2 = 0 \\ y[1] &= 0 - 0.5 \times 0 = 0 \\ y[2] &= 0 - 0.5 \times 0 = 0 \\ &\vdots \end{aligned}$$

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answer

(c) If $y[-1] = 0$ and $x[n] = \delta[n]$ then we can compute $y[n] = h[n]$, i.e., the impulse response. The corresponding transfer function is then from the equation for $Y(z)$:

$$H(z) = \mathcal{Z}[h[n]] = \frac{Y(z)}{X(z)} = \frac{1}{1 + 0.5z^{-1}}$$

If we want $y[n] = \delta[n] + 0.5\delta[n-1]$ or $Y(z) = 1 + 0.5z^{-1}$ then

$$X(z) = \frac{Y(z)}{H(z)} = (1 + 0.5z^{-1})^2 = 1 + z^{-1} + 0.25z^{-2}$$

which gives $x[n] = \delta[n] + \delta[n-1] + 0.25\delta(n-2)$

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Exercise 10

Inverse Z-transform and poles and zeros

When finding the inverse Z-transform of functions with z^{-1} terms in the numerator, the fact that z^{-1} can be thought of as a delay operator can be used to simplify the computation. Consider

$$X(z) = \frac{1 - z^{-10}}{1 - z^{-1}}$$

(a) Use the Z-transform of $u[n]$ and the properties of the Z-transform to find $x[n]$.

(b) If we consider $X(z)$ a polynomial in negative powers of z, what would be its degree and the values of its coefficients?

(c) Find the poles and the zeros of $X(z)$ and plot them on the z-plane. Is there a pole or zero at $z = 1$? Explain.

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answer

Pr 9.14 (a) Writing $X(z)$ as

$$X(z) = \frac{1}{1-z^{-1}} - \frac{z^{-10}}{1-z^{-1}}$$

since the inverse Z-transform of the first term is $u[n]$, then the inverse of the second is $-u[n-10]$ given that z^{-10} indicates a delay of 10 samples. Thus,

$$x[n] = u[n] - u[n-10] = \begin{cases} 1 & 0 \leq n \leq 9 \\ 0 & \text{otherwise} \end{cases}$$

(b) Although $X(z)$ has been shown as a ratio of two polynomials, using the above representation of $x[n]$ its Z-transform is

$$X(z) = 1 + z^{-1} + \cdots + z^{-9}$$

i.e., a 9^{th}–order polynomial in z^{-1}.
(c) We can rewrite $X(z)$ as

$$X(z) = \frac{z^{10}-1}{z^9(z-1)} = \frac{(z-1)\prod_{k=1}^{9}(z-e^{j\pi k/5})}{z^9(z-1)} = \frac{\prod_{k=1}^{9}(z-e^{j\pi k/5})}{z^9}$$

which is obtained by finding that the zeros of $X(z)$ are values $z_k^{10} = 1$ or $z_k = e^{j2\pi k/10}$ for $k = 0, \cdots, 9$. For $k = 0$ the zero is $z_0 = 1$, which cancels the pole at 1.

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Exercise 11

Initial conditions and impulse response

A second-order system has the difference equation

$$y[n] = 0.25y[n-2] + x[n]$$

where $x[n]$ is the input and $y[n]$ is the output.

(a) Find the input $x[n]$ so that for zero initial conditions, the output is given as $y[n] = 0.5^n u[n]$.

(b) If $x[n] = \delta[n] + 0.5\delta[n-1]$ is the input to the above difference equation, find the impulse response of the system.

Pr 9.16 (a) For zero initial conditions we want

$$Y(z) = \frac{X(z)}{1 - 0.25z^{-2}} = \frac{1}{1 - 0.5z^{-1}}$$

which gives that

$$X(z) = 1 + 0.5z^{-1} \quad \Rightarrow \quad x[n] = \delta[n] + 0.5\delta[n-1]$$

(b) For $x[n] = \delta[n] + 0.5\delta[n-1]$ we get $Y(z) = H(z)(1 + 0.5z^{-1})$ which from the difference equation is equal to

$$Y(z) = \frac{1 + 0.5z^{-1}}{1 - 0.25z^{-2}}$$

so that when comparing them we find that

$$H(z) = \frac{1 + 0.5z^{-1}}{(1 + 0.5z^{-1})(1 - 0.25z^{-2})} = \frac{1}{1 - 0.25z^{-2}} = \frac{1}{(1 - 0.5z^{-1})(1 + 0.5z^{-1})}$$

with poles at ± 0.5 so that

$$H(z) = \frac{A}{1 + 0.5z^{-1}} + \frac{B}{1 - 0.5z^{-1}}$$

where

$$\begin{aligned} A &= \frac{1}{1 - 0.5z^{-1}} |_{z^{-1}=-2} = 0.5 \\ B &= \frac{1}{1 + 0.5z^{-1}} |_{z^{-1}=2} = 0.5 \end{aligned}$$

so that

$$h[n] = 0.5(-0.5)^n u[n] + 0.5^{n+1} u[n]$$

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Exercise 12

Convolution sum and product of polynomials

The convolution sum is a fast way to find the coefficients of the polynomial resulting from the multiplication of two polynomials.

(a) Suppose $x[n] = u[n] - u[n-3]$. Find its Z-transform $X(z)$, a second-order polynomial in z^{-1}.

(b) Multiply $X(z)$ by itself to get a new polynomial $Y(z) = X(z)X(z) = X^2(z)$. Find $Y(z)$.

(c) Graphically show the convolution of $x[n]$ with itself and verify that the result coincides with the coefficients of $Y(z)$.

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answer

Pr 9.17 (a) The signal $x[n] = \delta[n] + \delta[n-1] + \delta[n-2]$ has a Z-transform

$$X(z) = 1 + z^{-1} + z^{-2}$$

(b) Then

$$Y(z) = X^2(z) = (1 + z^{-1} + z^{-2})^2 = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

The convolution of the coefficients of $X(z)$, or $x[n]$, with themselves gives the sequence

$$y[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

The length of $y[n]$ is twice that of $x[n]$ minus one, or $2 \times 3 - 1 = 5$ so that $Y(z)$ is a fourth -degree polynomial.

Exercise 13

Inverse Z-transform

Find the inverse Z-transform of

$$X(z) = \frac{8 - 4z^{-1}}{z^{-2} + 6z^{-1} + 8}$$

and determine $x[n]$ as $n \to \infty$. Assume $x[n]$ is causal.

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answer

<u>Pr 9.18</u> Writing $X(z)$ using terms found in tables, its partial fraction expansion is

$$\begin{aligned} X(z) &= \frac{2 - z^{-1}}{2(1 + 0.25z^{-1})(1 + 0.5z^{-1})} \\ &= \frac{A}{1 + 0.25z^{-1}} + \frac{B}{1 + 0.5z^{-1}} \end{aligned}$$

corresponding to the poles at -0.25 and -0.5. The coefficients of the expansion are

$$\begin{aligned} A &= \frac{2 - z^{-1}}{2(1 + 0.5z^{-1})} |_{z^{-1}=-4} = -3 \\ B &= \frac{2 - z^{-1}}{2(1 + 0.25z^{-1})} |_{z^{-1}=-2} = 4 \end{aligned}$$

so that

$$X(z) = \frac{-3}{1 + 0.25z^{-1}} + \frac{4}{1 + 0.5z^{-1}}$$

and the inverse is

$$x[n] = [-3(-0.25)^n + 4(-0.5)^n]u[n]$$

and in the steady–state it is zero.

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Exercise 14

Z-transform properties and inverse transform

Sometimes the partial fraction expansion is not needed in finding the inverse Z-transform—instead the properties of the transform can be used. Consider the function

$$F(z) = \frac{z+1}{z^2(z-1)}$$

(a) Determine whether $F(z)$ is a proper rational function as a function of z and of z^{-1}.

(b) Verify that $F(z)$ can be written as

$$F(z) = \frac{z^{-2}}{1-z^{-1}} + \frac{z^{-3}}{1-z^{-1}}$$

Find the inverse Z-transform $f[n]$ using the above expression.

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answer

Pr 9.19 (a) $F(z)$ is a proper rational function in positive powers of z as its numerator is of lower order than the denominator. If we convert it into negative powers, z^{-1}, we have

$$F(z) = \frac{z^{-2}(1+z^{-1})}{1-z^{-1}}$$

which is not proper rational in z^{-1} as its numerator is of higher order than its denominator.
(b) Using the above expression we have that

$$F(z) = \frac{z^{-2}}{1-z^{-1}} + \frac{z^{-3}}{1-z^{-1}}$$

which gives

$$f[n] = u[n-2] + u[n-3]$$

given that $1/(1-z^{-1})$ is the Z-transform of $u[n]$ and z^{-2} and z^{-3} delay $u[n]$ by 2 and 3 samples.

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Exercise 15

partial fraction expansion

(a) Find the inverse Z-transform of $a/(1 - az^{-1})^2$.

(b) Suppose that the partial fraction expansion given by MATLAB is

$$X(z) = \frac{-1}{1 - 0.5z^{-1}} + \frac{1}{(1 - 0.5z^{-1})^2}$$

Determine the inverse $x[n]$.

Exercises Ch.10 z-transform

answer

<u>Pr 9.23</u> (a) In the Z-transform table we find the pair

$$na^n u[n] \;\Leftrightarrow\; \frac{az^{-1}}{(1-az^{-1})^2}.$$

To obtain $a/(1-az^{-1})^2$ we multiply the above Z-transform by z, i.e., in the time-domain we advance the signal by one to get $(n+1)a^{n+1}u[n+1] = (n+1)a^{n+1}u[n]$, since at $n=-1$ we get that $n+1=0$. Thus the pair

$$(n+1)a^{n+1}u[n] \;\Leftrightarrow\; \frac{a}{(1-az^{-1})^2}.$$

(b) The given $X(z)$ equals

$$X(z) = \frac{-(1-0.5z^{-1})+1}{(1-0.5z^{-1})^2} = \frac{0.5z^{-1}}{(1-0.5z^{-1})^2}$$

which as indicated at the beginning of the previous part corresponds to $x[n] = 0.5^n nu[n]$. From the partial expansion we have using the second pair

$$x[n] = -0.5^n u[n] + 2(n+1)0.5^{(n+1)}u[n] = 0.5^n(-1+2\times 0.5(n+1))u[n] = 0.5^n nu[n]$$