Exercises Ch.8 Sampling

Exercise 1 – exercises taken from Chaparro (first edition)

Sampling actual signals

Consider the sampling of real signals.

(a) Typically, a speech signal that can be understood over a telephone shows frequencies from about 100 Hz to about 5 KHz. What would be the sampling frequency f_s (samples/sec) that would be used to sample speech without aliasing? How many samples would you need to save when storing an hour of speech? If each sample is represented by 8 bits, how many bits would you have to save for the hour of speech?

(b) A music signal typically displays frequencies from 0 up to 22 KHz. What would be the sampling frequency f_s that would be used in a CD player?

(c) If you have a signal that combines voice and musical instruments, what sampling frequency would you use to sample this signal? How would the signal sound if played at a frequency lower than the Nyquist sampling frequency?

Exercises Ch.8 Sampling

answer

<u>Pr 7.1</u> (a) The maximum frequency in a speech signal is $f_{max} = 5$ KHz, so that its sampling frequency should be

$$f_s \geq 2f_{max} = 10\text{KHz or } 10,000 \text{ samples/sec}$$

The number of samples in an hour of sampling speech is

$$\text{Samples/hour } \geq 3,600 \text{ sec/hour } \times 10,000 \text{ samples/sec } = 3.6 \times 10^7 \text{samples/hour}$$

and the number of bits is

$$\text{Bits/hour } \geq 3.6 \times 10^7 \text{samples/hour} \times 8 \text{ bits/sample } = 288 \times 10^6 \text{ bits/hour}$$

(b) Since $f_{max} = 22$ KHz can be considered the maximum frequency in a music signal, then the sampling frequency should be

$$f_s \geq 2f_{max} = 44 \text{ KHz}$$

(c) We need to use the higher of the above sampling frequencies to accommodate both signals, so $f_s \geq 44$ KHz

Exercises Ch.8 Sampling

Exercise 2

Sampling of band-limited signals

Consider the sampling of a sinc signal and related signals.

(a) For the signal $x(t) = \sin(t)/t$, find its magnitude spectrum $|X(\Omega)|$ and determine if this signal is band limited or not.

(b) Suppose you want to sample $x(t)$). What would be the sampling period T_s you would use for the sampling without aliasing?

(c) For a signal $y(t) = x^2(t)$, what sampling frequency f_s would you use to sample it without aliasing? How does this frequency relate to the sampling frequency used to sample $x(t)$?

(d) Find the sampling period T_s to sample $x(t)$ so that the sampled signal $x_s(0) = 1$, otherwise $x_s(nT_s) = 0$ for $n \neq 0$.

Exercises Ch.8 Sampling

answer

<u>Pr 7.2</u> (a) To find $X(\Omega)$, we use duality or find the inverse Fourier transform of a pulse of amplitude A and bandwidth Ω_0, that is

$$X(\Omega) = A[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)]$$

so that

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} A e^{j\Omega t} d\Omega = \frac{A}{2\pi j t} e^{j\Omega t}\Big|_{-\Omega_0}^{\Omega_0} \\ &= \frac{A}{\pi t}\sin(\Omega_0 t) \end{aligned}$$

which when compared with the given $x(t) = \sin(t)/t$ gives that $A = \pi$ and $\Omega_0 = 1$ or

$$X(\Omega) = \pi[u(\Omega + 1) - u(\Omega - 1)]$$

indicating that $x(t)$ is band-limited with a maximum frequency $\Omega_{max} = 1$ (rad/sec).
(b) To sample without aliasing the sampling frequency should be chosen to be

$$f_s = \frac{1}{T_s} \geq 2\frac{\Omega_{max}}{2\pi}$$

which gives a sampling period

$$T_s \leq \frac{\pi}{\Omega_{max}} = \pi \ \text{sec/sample}$$

Exercises Ch.8 Sampling

(c) The spectrum of $y(t) = x^2(t)$ is the convolution in the frequency

$$Y(\Omega) = \frac{1}{2\pi} X(\Omega) * X(\Omega)$$

which would have a maximum frequency $\Omega_{max} = 2$, giving a sampling frequency which is double the one for $x(t)$. The sampling period for $y(t)$ should be

$$T_s \leq \frac{\pi}{2}.$$

(d) The signal $x(t) = \sin(t)/t$ is zero whenever $t = \pm k\pi$, for $k = 1, 2, \cdots$ so that choosing $T_s = \pi$ (the Nyquist sampling period) we obtain the desired signal $x_s(0) = 1$ and $x(nT_s) = 0$.

Exercises Ch.8 Sampling

Exercise 3

Sampling of time-limited signals—MATLAB

Consider the signals $x(t) = u(t) - u(t-1)$ and $y(t) = r(t) - 2r(t-1) + r(t-2)$.

(a) Are either of these signals band limited? Explain.

(b) Use Parseval's theorem to determine a reasonable value for a maximum frequency for these signals (choose a frequency that would give **90%** of the energy of the signals). Use MATLAB.

(c) If we use the sampling period corresponding to $y(t)$ to sample $x(t)$, would aliasing occur? Explain.

(d) Determine a sampling period that can be used to sample both $x(t)$ and $y(t)$ without causing aliasing in either signal.

Exercises Ch.8 Sampling

answer

Pr 7.3 (a) Because both of these signals have finite time support they are not band-limited, their support in frequency is infinite. For $x(t)$, its Fourier transform is

$$\begin{aligned} X(\Omega) &= \frac{1}{s}(1-e^{-s})|_{s=j\Omega} = \frac{1}{j\Omega}e^{-j\Omega/2}2j\sin(\Omega/2) \\ &= \frac{\sin(\Omega/2)}{\Omega/2}e^{-j\Omega/2} \end{aligned}$$

For $y(t)$ we have that

$$\frac{dy(t)}{dt} = x(t) - x(t-1) = u(t) - 2u(t-1) + u(t-2)$$

so that

$$\begin{aligned} j\Omega Y(\Omega) &= \frac{1}{s}e^{-s}(e^{s} - 2 + e^{-s})\,|_{s=j\Omega} = \frac{2}{j\Omega}e^{-j\Omega}(\cos(\Omega)-1) \\ &= \frac{4}{j\Omega}e^{-j\Omega}\sin^2(\Omega/2) \end{aligned}$$

using $-\sin^2(\theta) = \frac{1}{2}(\cos(2\theta)-1)$. So that

$$Y(\Omega) = e^{-j\Omega}\left(\frac{\sin(\Omega/2)}{\Omega/2}\right)^2$$

Exercises Ch.8 Sampling

(b) The energy of $x(t)$ is unity (the area under $|x(t)|^2$), i.e., $E_x = 1$. According to Parseval, 90% of the energy of $x(t)$ is within the frequency band $[-\Omega_0, \Omega_0]$ or

$$\begin{aligned} 0.9 &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} |X(\Omega)|^2 d\Omega \\ &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} \left|\frac{\sin(\Omega/2}{\Omega/2}\right|^2 d\Omega \end{aligned}$$

Using MATLAB symbolic computation we find the integral gives that $\Omega_0 = 5.6$ (rad/sec).

For $y(t)$, the energy is

$$E_y = 2\int_0^1 t^2 dt = 2\frac{t^3}{3}\,|_{t=0}^1 = \frac{2}{3}$$

Ninety percent of the energy for this signal is in the frequency band $[-\Omega_1, \Omega_1]$ or

$$\begin{aligned} 0.9\,\frac{2}{3} &= \frac{1}{2\pi}\int_{-\Omega_1}^{\Omega_1} |X(\Omega)|^4 d\Omega \\ &= \frac{1}{2\pi}\int_{-\Omega_1}^{\Omega_1} \left|\frac{\sin(\Omega/2}{\Omega/2}\right|^4 d\Omega \end{aligned}$$

because $|Y(\Omega)| = |X(\Omega)|^2$.

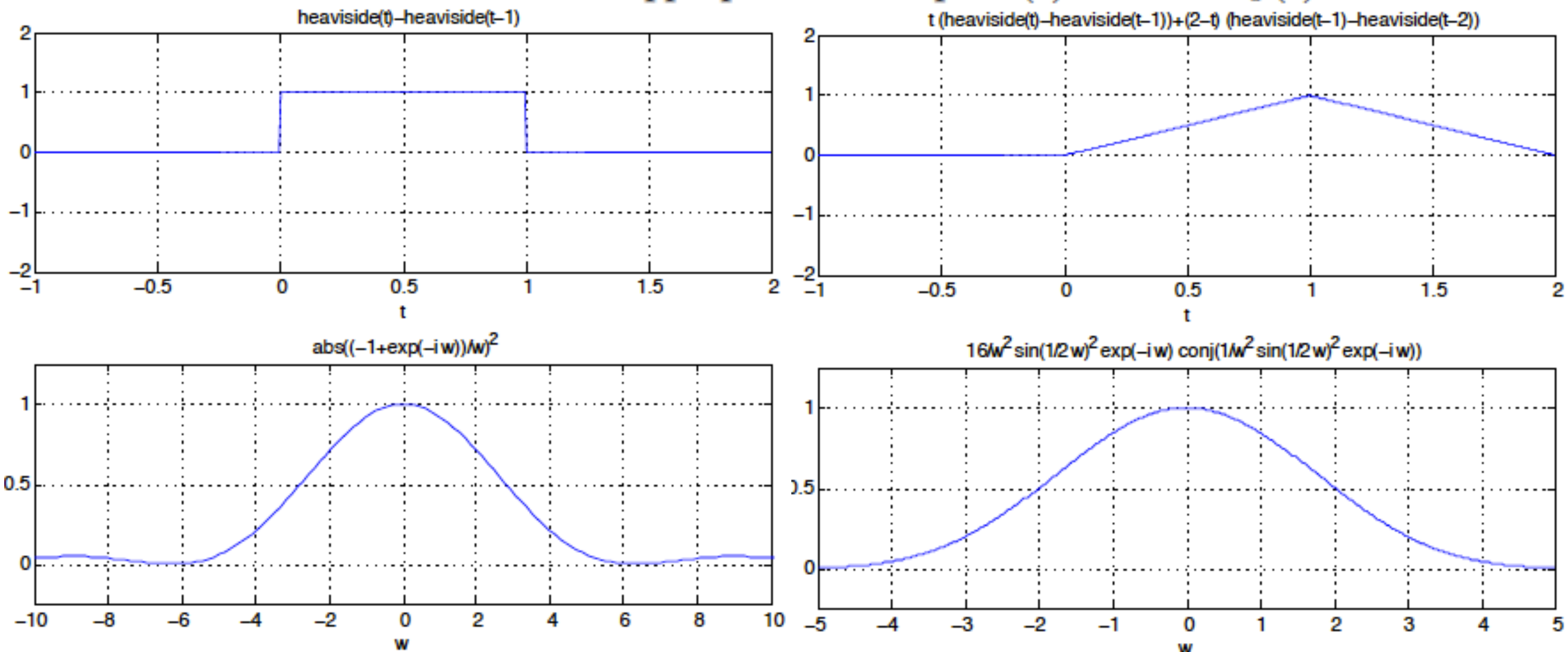
Again, we use MATLAB symbolic computation to find the value of Ω_1.

Exercises Ch.8 Sampling

This script gives that $\Omega_1 = 2.6$ (rad/sec) $< \Omega_0$. (c) The signal $y(t)$ is smoother than $x(t)$ (this signal displays higher frequencies than $y(t)$) so that we expect that $\Omega_{max\ y} < \Omega_{max\ x}$. Accordingly the sampling period T_{sy} to sample without aliasing $y(t)$ is larger than the one required to sample $x(t)$. If we choose as maximum frequencies the values Ω_0 and Ω_1 for the signals $x(t)$ and $y(t)$ we calculated above, we should have that $\Omega_0 > \Omega_1$ and choosing the Nyquist sampling periods (the values that give an equality in the Nyquist sampling rate condition)

$$T_{sy} = \frac{\pi}{\Omega_1} > \frac{\pi}{\Omega_0} = T_{sx}$$

Thus aliasing would be caused by using T_{sy} to sample $x(t)$.
(d) On the other hand, T_{sx} would be appropriate to sample $x(t)$ as well as $y(t)$.



$x(t)$ *and* $|X(\Omega)|^2$. $y(t)$ *and* $|Y(\Omega)|^2$.

Exercises Ch.8 Sampling

Exercise 4

Nyquist sampling rate condition and aliasing
Consider the signal

$$x(t) = \frac{\sin(0.5t)}{0.5t}$$

(a) Find the Fourier transform $X(\Omega)$ of $x(t)$.
(b) Is $x(t)$ band limited? If so, find its maximum frequency Ω_{max}.
(c) Suppose that $T_s = 2\pi$. How does Ω_s relate to the Nyquist frequency $2\Omega_{max}$? Explain.
(d) What is the sampled signal $x(nT_s)$ equal to? Carefully plot it and explain if $x(t)$ can be reconstructed.

Exercises Ch.8 Sampling

answer

Pr. 7.5 (a) The Fourier transform is

$$X(\Omega) = 2\pi[u(\Omega + 0.5) - u(\Omega - 0.5)]$$

(b) $x(t)$ is clearly band-limited with $\Omega_{max} = 0.5$ (rad/sec).

(c) According to the Nyquist sampling rate condition, we should have that

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_{max}$$

or the sampling period

$$T_s \leq \frac{\pi}{\Omega_{max}} = 2\pi$$

The given value satisfies the Nyquist sampling rate condition so we can sample the signal with no aliasing. The given sampling period is the Nyquist sampling period.

Exercises Ch.8 Sampling

(d) Plotting the sinc function it can be seen that it is zero at values of $0.5t = \pm\pi k$ or $t = \pm 2\pi k$ for an integer k. The sampled signal using $T_s = 2\pi$ is

$$x(nT_s) = \frac{\sin(0.5\,2\pi n)}{0.5n\,2\pi} = \frac{\sin(\pi n)}{\pi n}$$

which is 1 for $n = 0$, and 0 for any other value of n. It seems the signal cannot be reconstructed from the samples, that frequency aliasing has occurred. Ideally, that is not the case. The spectrum of the sampled signal $x_s(t)$ for $T_s = 2\pi$ ($\Omega_s = 1$) is

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(\Omega - k) = 1$$

ignoring the discontinuities at frequencies $\pm 0.5k$ caused by the overlap of the spectra. Passing this signal through an ideal low-pass filter with amplitude 2π and cut-off frequency $\Omega_s/2 = 1/2$ the reconstructed signal, the output of this filter, is the inverse Fourier transform of a pulse in frequency, i.e., a sinc function, that coincides with the original signal.

Exercises Ch.8 Sampling

Exercise 5

Anti-aliasing

Suppose you want to find a reasonable sampling period T_s for the noncausal exponential

$$x(t) = e^{-|t|}$$

(a) Find the Fourier transform of $x(t)$, and plot $|X(\Omega)|$. Is $x(t)$ band limited?

(b) Find a frequency Ω_0 so that 99% of the energy of the signal is in $-\Omega_o \leq \Omega \leq \Omega_o$.

(c) If we let $\Omega_s = 2\pi/T_s = 5\Omega_0$, what would be T_s?

(d) Determine the magnitude and bandwidth of an anti-aliasing filter that would change the original signal into the band-limited signal with 99% of the signal energy.

Exercises Ch.8 Sampling

answer

<u>Pr 7.6</u> (a) The Fourier transform is

$$X(\Omega) = \frac{2}{1-s^2}\,|_{s-j\Omega} = \frac{2}{1+\Omega^2}$$

$x(t)$ is not band-limited, the frequency support of $X(\Omega)$ is infinite.

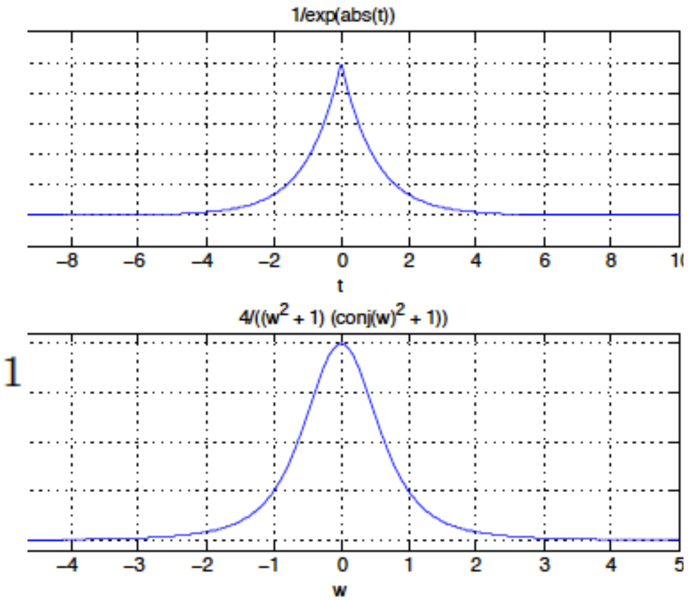
(b) The energy of the signal is

$$E_x = 2\int_0^{\infty} e^{-2t}dt = 2\frac{e^{-2t}}{-2}\,|_0^{\infty} = 1$$

thus 99% of the energy in the frequency domain is

$$0.99 = \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} \left|\frac{2}{1+\Omega^2}\right| d\Omega$$

for some value of Ω_0 which can be obtained using the following script:



Signal $x(t)$ *and* $|X(\Omega)|^2$

The maximum frequency is found to be 3.3 (rad/sec).

(d) For a sampling frequency $\Omega_s = 5\Omega_0 = 16.5$ the sampling period would be $T_s = 2\pi/\Omega_s = 2\pi/16.5 = 0.3808$ sec/sample.

(e) Looking at the spectrum $|X(\Omega)|$, an anti-aliasing filter would be an ideal low-pass filter with a magnitude of 1 and a cut-off frequency $\Omega_c = \Omega_0 = 3.3$ (rad/sec).

Exercise 6

Sampling of modulated signals

Assume you wish to sample an amplitude modulated signal

$$x(t) = m(t)\cos(\Omega_c t)$$

where $m(t)$ is the message signal and $\Omega_c = 2\pi\,10^4$ rad/sec is the carrier frequency.

(a) If the message is an acoustic signal with frequencies in a band of $[0, 22]$ KHz, what would be the maximum frequency present in $x(t)$?

(b) Determine the range of possible values of the sampling period T_s that would allow us to sample $x(t)$ satisfying the Nyquist sampling rate condition.

(c) Given that $x(t)$ is a band-pass signal, compare the above sampling period with the one that can be used to sample band-pass signals.

answer

Pr 7.7 (a) The Fourier transform of $x(t)$ is

$$X(\Omega) = 0.5M(\Omega - \Omega_c) + 0.5M(\Omega + \Omega_c)$$

where $M(\Omega)$ is the Fourier transform of $m(t)$. The maximum frequency present in $x(t)$ is

$$\Omega_{max} = \Omega_c + 2\pi \times 22 \times 10^3 = 2\pi(10 + 22)10^3 = 64\pi\ 10^3$$

(b) The sampling frequency is

$$\Omega_s = \frac{2\pi}{T_s} \geq 2 \times 64\pi \times 10^3 = 128 \times 10^3$$

so that

$$T_s \leq \frac{1}{64}\ 10^{-3} \text{ sec/sample}$$

(c) The bandwidth of the message is $B = 2\pi \times 22 \times 10^3 = 44\pi \times 10^3$ rad/sec, using this frequency $x(t)$ can be sampled at $2B = 88\pi \times 10^3$ rad/sec which is much smaller than the one found above.

Exercises Ch.8 Sampling

Exercise 7

Sampling output of nonlinear system

The input–output relation of a nonlinear system is

$$y(t) = x^2(t)$$

where $x(t)$ is the input and $y(t)$ is the output.

(a) The signal $x(t)$ is band limited with a maximum frequency $\Omega_M = 2000\pi$ rad/sec. Determine if $y(t)$ is also band limited, and if so, what is its maximum frequency $\Omega_{\max}$?

(b) Suppose that the signal $y(t)$ is low-pass filtered. The magnitude of the low-pass filter is unity and the cut-off frequency is $\Omega_c = 5000\pi$ rad/sec. Determine the value of the sampling period T_s according to the given information.

(c) Is there a different value for T_s that would satisfy the Nyquist sampling rate condition for both $x(t)$ and $y(t)$ and that is larger than the one obtained above? Explain.

answer

Pr 7.8 (a) If the signal $x(t)$ is band-limited $y(t) = x^2(t)$ has a Fourier transform $Y(\Omega) = (1/2\pi)(X(\Omega) * X(\Omega))$ having a bandwidth double that of $x(t)$, or

$$\Omega_{max\ y} = 2\Omega_M = 4000\pi$$

(b) Filtering with a low-pass filter of cut-off frequency 5000π would not change the maximum frequency of $Y(\Omega)$ so that

$$T_s \leq \frac{\pi}{4000\pi} = \frac{1}{4000} = 0.25 \times 10^{-3}$$

(c) No. For $x(t)$,

$$T_{s1} \leq \frac{\pi}{2000\pi} = 0.5 \times 10^{-3}$$

and if $T_s = 0.25 \times 10^{-3}$ then $T_{s1} \leq 2T_s$, so that we need to use T_s to sample both $x(t)$ and $y(t)$.

Exercises Ch.8 Sampling

Exercise 8

Signal reconstruction

You wish to recover the original analog signal $x(t)$ from its sampled form $x(nT_s)$.

(a) If the sampling period is chosen to be $T_s = 1$ so that the Nyquist sampling rate condition is satisfied, determine the magnitude and cut-off frequency of an ideal low-pass filter $H(j\Omega)$ to recover the original signal and plot them.

(b) What would be a possible maximum frequency of the signal? Consider an ideal and a nonideal low-pass filter to reconstruct $x(t)$. Explain.

answer

<u>Pr 7.9</u> (a) If $T_s = 1$ then

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_{max}$$

or $\Omega_{max} \leq \pi$. To reconstruct the original signal we choose the cutoff frequency of the ideal low-pass filter to be

$$\Omega_{max} < \Omega_c < 2\pi - \Omega_{max}$$

and the magnitude $T_s = 1$.

(b) Since $T_s \leq \pi/\Omega_{max}$, and $T_s = 1$ then $\Omega_{max} \leq \pi$. If $\Omega_{max} = \pi$ for an ideal low-pass filter, then $\Omega_c = \pi$ to recover the original signal. Thus the maximum frequency has to be smaller than π to make it possible to use an ideal or a non-ideal low-pass filter to recover the original signal.