Exercises Ch.9 Discrete-time LTI systems

Exercise 1 – exercises taken from Chaparro (first edition)

Periodicity of sampled signals—MATLAB

Consider an analog periodic sinusoid $x(t) = \cos(3\pi t + \pi/4)$ being sampled using a sampling period T_s to obtain the discrete-time signal $x[n] = x(t)|_{t=nT_s} = \cos(3\pi T_s n + \pi/4)$.

(a) Determine the discrete frequency of $x[n]$.

(b) Choose a value of T_s for which the discrete-time signal $x[n]$ is periodic. Use MATLAB to plot a few periods of $x[n]$, and verify its periodicity.

(c) Choose a value of T_s for which the discrete-time signal $x[n]$ is not periodic. Use MATLAB to plot $x[n]$ and choose an appropriate length to show the signal is not periodic.

(d) Determine under what condition the value of T_s makes $x[n]$ periodic.

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answer

Pr 8.3 (a) The discrete-time signal is

$$x[n] = x(nT_s) = \cos(3\pi T_s n + \pi/4) = A\cos(\omega_0 n + \pi/4)$$

so that its magnitude is $A = 1$ and its discrete frequency $\omega_0 = 3\pi T_s$ (rad).
(b) Any value for which

$$3\pi T_s = \frac{2\pi m}{N}$$

where m and N are non-divisible positive integers, or

$$T_s = \frac{2m}{3N}$$

would make $x(nT_s)$ periodic of period N. Suppose for instance that we let $N = 2$ and $m = 1$, then $T_s = 1/3$ would satisfy the condition, giving $x[n] = x(nT_s) = \cos(3\pi n/3 + \pi/4) = \cos(2\pi n/2 + \pi/4)$, of period $N = 2$.
(c) If

$$3\pi T_s \neq \frac{2\pi m}{N}$$

then the corresponding value of T_s would give a non-periodic discrete-time signal. For instance, if

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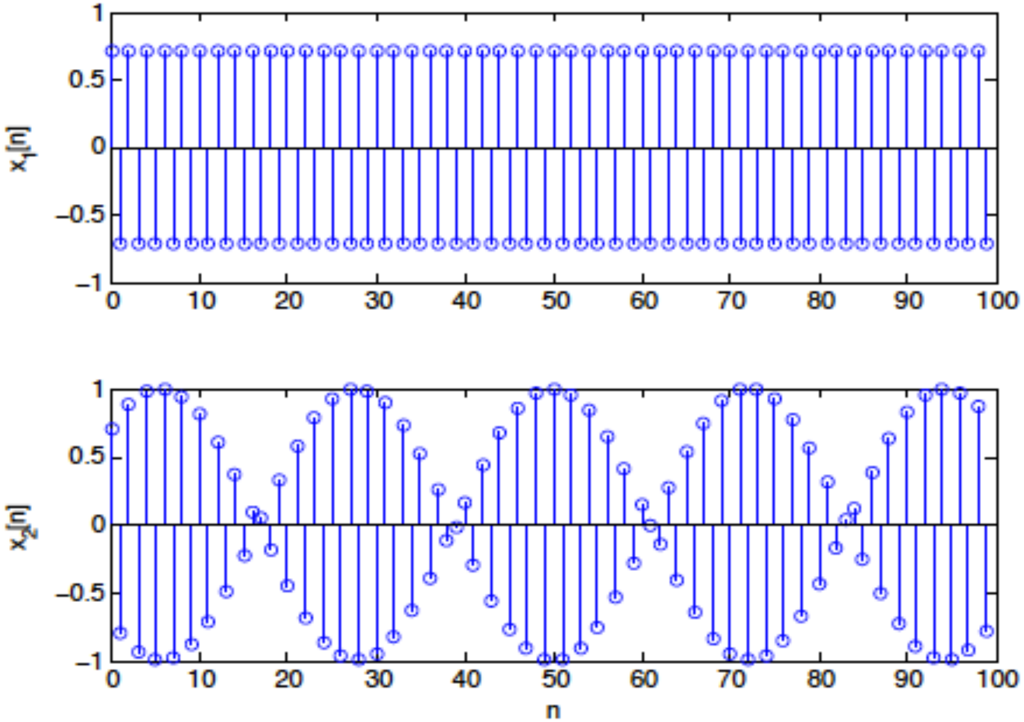
$$T_s = \frac{2m}{3\pi N}$$

and again let $N = 2$, $m = 3$ to get an irrational value $T_s = 1/\pi$, the discrete-time signal is $x[n] = x(nT_s) = \cos(3n + \pi/4)$ which is not periodic.

(d) The condition is that

$$T_s = \frac{2m}{3N}$$

be a rational number where m and N are non-divisible positive integers.



Periodic signal $x_1[n]$ (top), non-periodic $x_2[n]$.

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Exercise 2

Even and odd decomposition and energy—MATLAB

Suppose you sample the analog signal

$$x(t) = \begin{cases} 1 - t & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

with a sampling period $T_s = 0.25$ to generate $x[n] = x(t)|_{t=nT_s}$.

(a) Use MATLAB to plot $x[-n]$ for an appropriate interval.

(b) Find $x_e[n] = 0.5[x[n] + x[-n]]$ and plot it carefully using MATLAB.

(c) Find $x_o[n] = 0.5[x[n] - x[-n]]$ and plot it carefully using MATLAB.

(d) Verify that $x_e[n] + x_o[n] = x[n]$ graphically.

(e) Compute the energy of $x[n]$ and compare it to the sum of the energies of $x_e[n]$ and $x_o[n]$.

answer

Pr 8.4 The discrete-time signal obtained from sampling $x(t)$ using $T_s = 0.25$ is

$$x[n] = \begin{cases} 1 - 0.25n & 0 \leq 0.25n \leq 1 \text{ or } 0 \leq n \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

(a) The reflected signal $x[-n] = 1 + 0.25n$ for $-4 \leq n \leq 0$ and zero otherwise.

(b) The signal $x[n]$ is neither even nor odd, so its even component is

$$x_e[n] = \frac{1}{2}[x[n] + x[-n]]$$

which can be written as

$$x_e[n] = \begin{cases} (1 - 0.25n)/2 & 1 \leq n \leq 4 \\ 1 & n = 0 \\ (1 + 0.25n)/2 & -4 \leq n \leq -1 \end{cases}$$

(c) The odd component of $x[n]$ is given by

$$x_o[n] = \frac{1}{2}[x[n] - x[-n]]$$

which can be written as

$$x_o[n] = \begin{cases} (1 - 0.25n)/2 & 1 \leq n \leq 4 \\ 0 & n = 0 \\ -(1 + 0.25n)/2 & -4 \leq n \leq -1 \end{cases}$$

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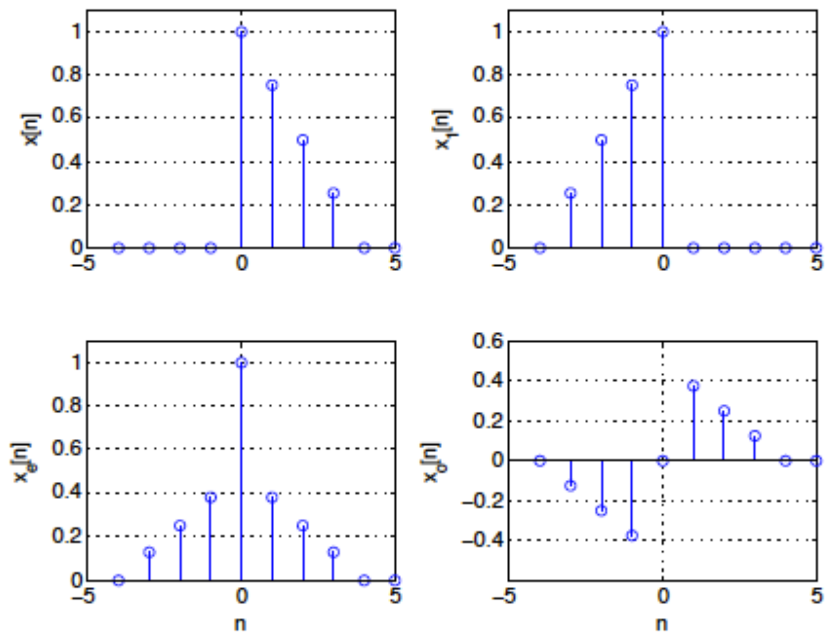
(d) Clearly adding the even and the odd components gives $x[n]$.

(e) The energy of the signal $x[n]$ can be written in terms of the energy of the even and the odd components. Since $x[n] = 0$ for $n < 0$ then

$$\sum_{n=0}^{4} x^2[n] = \sum_{n=-4}^{4} x^2[n]$$
$$= \sum_{n=-4}^{4} [x_e^2[n] + x_o^2[n] + 2x_e[n]x_o[n]]$$

the last term is zero, given that the signal $x_e[n]x_o[n]$ is odd and is being computed over $-4 \leq n \leq 4$. So we have that

$$\varepsilon_x^2 = \varepsilon_{x_e}^2 + \varepsilon_{x_o}^2$$



Exercise 3

Absolutely summable and finite-energy discrete-time signals—MATLAB

Suppose we sample the analog signal $x(t) = e^{-2t}u(t)$ using a sample period $T_s = 1$.

(a) Expressing the sampled signal as $x(nT_s) = x[n] = \alpha^n u[n]$, what is the corresponding value of α? Use MATLAB to plot $x[n]$.

(b) Show that $x[n]$ is absolutely summable—that is, show the following sum is finite:

$$\sum_{n=-\infty}^{\infty} |x[n]|$$

(c) If you know that $x[n]$ is absolutely summable, could you say that $x[n]$ is a finite-energy signal? Use MATLAB to plot $|x[n]|$ and $x^2[n]$ in the same plot to help you decide.

(d) In general, for what values of α are signals $y[n] = \alpha^n u[n]$ finite energy? Explain.

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answer

Pr 8.8 (a) The sampled signal is given by

$$x(nT_s) = x[n] = e^{-2n}u[n] = (e^{-2})^n u[n]$$

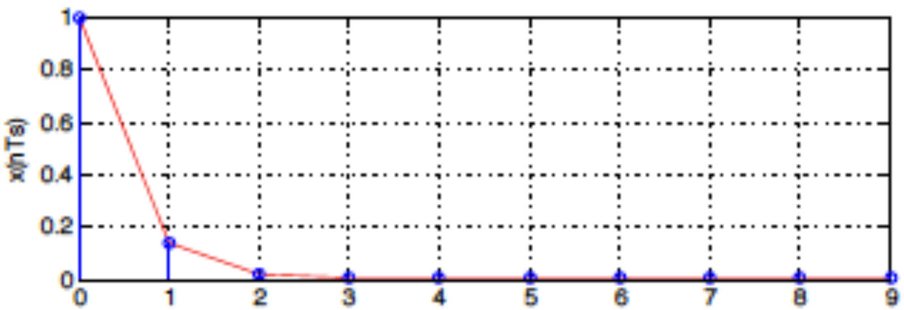
which indicates that $\alpha = 1/e^2 < 1$.
(b) The sum

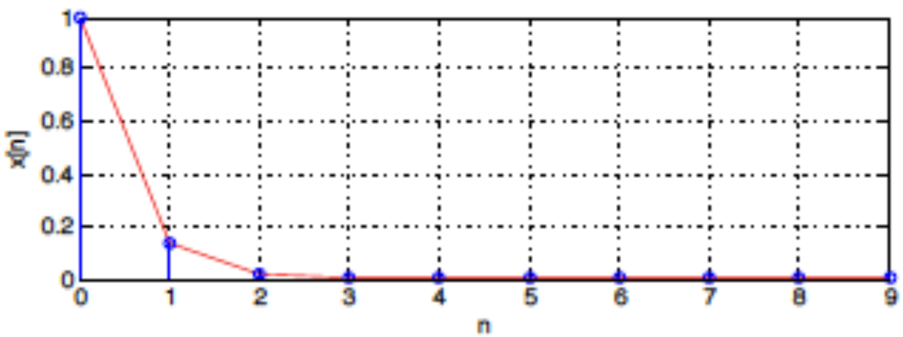
$$\sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} \qquad |\alpha| < 1$$

or $e^2/(e^2-1)$ is finite, so $x[n]$ is absolutely summable.
(c) Given that $\alpha^n > \alpha^{2n}$, because $\alpha < 1$, we then have that if $x[n]$ is absolutely summable it is finite energy. Indeed,

$$\sum_{n=0}^{\infty} (\alpha^2)^n = \frac{1}{1-\alpha^2} = \frac{e^4}{e^4-1} < \infty$$

so the signal $x[n]$ is finite energy.





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Exercise 4

Discrete-time periodic signals

Determine whether the following discrete-time sinusoids are periodic or not. If periodic, determine its period N_0.

$$x[n] = 2\cos(\pi n - \pi/2)$$

$$y[n] = \sin(n - \pi/2)$$

Periodicity of discrete-time signals

Consider periodic signals $x[n]$, of period $N_1 = 4$, and $y[n]$, of period $N_2 = 6$. What would be the period of

$$z[n] = x[n] + y[n]$$

$$v[n] = x[n]y[n]$$

$$w[n] = x[2n]$$

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Pr 8.9 The discrete frequency for the given signals are

$$
\begin{aligned}
x[n]: &\quad \omega_0 = \pi = \frac{2\pi}{2} \quad \Rightarrow \quad \text{periodic with period } \; N_0 = 2 \\
y[n]: &\quad \omega_0 = 1 \neq \frac{2\pi m}{N_0}, \text{ not periodic} \\
z[n]: &\quad \text{not periodic, as } y[n] \text{ is not perioidic} \\
v[n]: &\quad \omega_0 = \frac{3\pi}{2} = \frac{2\pi}{4}3, \quad \Rightarrow \quad \text{periodic with period } \; N_0 = 4
\end{aligned}
$$

Pr 8.10 $x[n]$ is periodic of period $N_1 = 4$, and $y[n]$ is periodic of period $N_2 = 6$ so that

$$\frac{N_1}{N_2} = \frac{4}{6} = \frac{2}{3}$$

then the sum $z[n] = x[n] + y[n]$ is periodic of period $3N_1 = 2N_2 = 12$, i.e., three periods of $x[n]$ fit in 2 of $y[n]$.

Similarly, $v[n]$ is periodic of period 12. Indeed,

$$v[n+12] = x[n+12]y[n+12] = x[n]y[n]$$

since 12 is three times the period of $x[n]$ and two times the period of $y[n]$.

The compressed signal $w[n] = x[2n]$ has period $N_1/2 = 2$, since

$$w[n+2] = x[2(n+2)] = x[2n+4] = x[2n] = w[n]$$

since $x[n]$ is periodic of period 4.

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Exercise 5

LTI of ADCs

An ADC can be thought of as composed of three subsystems: a sampler, a quantizer, and a coder.

(a) The sampler, as a system, has as input an analog signal $x(t)$ and as output a discrete-time signal $x(nT_s) = x(t)|_{t=nT_s}$ where T_s is the sampling period. Determine whether the sampler is a linear system or not.

(b) Sample $x(t) = \cos(0.5\pi t)u(t)$ and $x(t-0.5)$ using $T_s = 1$ to get $y(nT_s)$ and $z(nT_s)$, respectively. Plot $x(t)$, $x(t-0.5)$, and $y(nT_s)$ and $z(nT_s)$. Is $z(nT_s)$ a shifted version of $y(nT_s)$ so that you can say the sampler is time invariant? Explain.

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answer

Pr 8.14 (a) The sampler is linear: if $x_i(t), i = 1, 2$, are the inputs then $x_i(nT_s), i = 1, 2$ are the outputs. When the input is $ax_1(t) + bx_2(t)$, a combination of the previous two inputs, then the output will be $ax_1(nT_s) + bx_2(nT_s)$ or the combination of the previous outputs.
(b) For $T_s = 1$, the sampling of $x(t) = \cos(0.5\pi t)u(t)$ gives the signal

$$y(nT_s) = \cos(0.5\pi n)u[n]$$

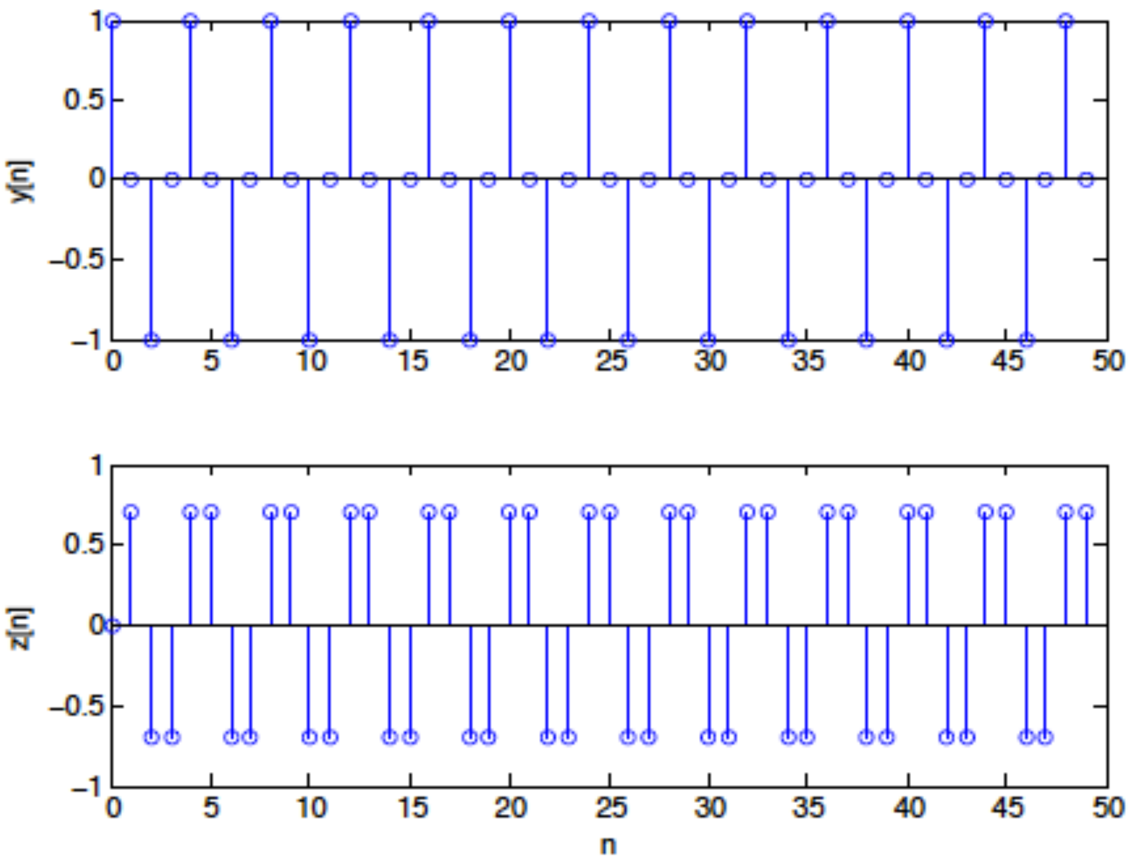
while for $T_s = 1$ the sampling of $x(t - 0.5) = \cos(0.5\pi(t - 0.5))u(t - 0.5)$ gives

$$z(nT_s) = \cos(0.5\pi(n - 0.5))u(n - 0.5)$$

Let us compare the first five values

$$\begin{array}{ll} y(0) = 1 & z(0) = 0 \\ y(1) = 0 & z(1) = \cos(0.25\pi) = 0.707 \\ y(2) = -1 & z(2) = \cos(0.75\pi) = -0.707 \\ y(3) = 0 & z(3) = \cos(1.25\pi) = -0.707 \\ y(4) = 1 & z(4) = \cos(1.75\pi) = 0.707 \end{array}$$

showing clearly that $z(nT_s)$ is not $y(nT_s)$ shifted. Thus the sampler is time-varying. Notice that if we sample the analog signal shifted by exactly one or more sampling periods, the output signal is a shifted version of the originally sampled signal. But if the shift is not a multiple of the sampling period, it is not. To verify that plot $y[n]$ and $z[n]$ for $0 \leq n \leq 49$.



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Exercise 6

Impulse response of an IIR system—MATLAB

A discrete-time IIR system is represented by the difference equation

$$y[n] = 0.15y[n-2] + x[n] \quad n \geq 0$$

where $x[n]$ is the input and $y[n]$ is the output.

(a) To find the impulse response $h[n]$ of the system, let $x[n] = \delta[n]$, $y[n] = h[n]$, and the initial conditions be zero, $y[n] = h[n] = 0$, $n < 0$. Find recursively the values of $h[n]$ for values of $n \geq 0$.

(b) As a second way to do it, replace the relation between the input and the output given by the difference equation to obtain a convolution sum representation that will give the impulse response $h[n]$. What is $h[n]$?

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answer

Pr 8.17 (a) The impulse response can be found solving the difference equation recursively, i.e., letting the output be $y[n] = h[n]$ and the input be $x[n] = \delta[n]$ and assuming zero initial conditions, i.e., $h[n] = 0$ for $n < 0$. We have then

$$h[n] = 0.15h[n-2] + \delta[n]$$

which for $n \geq 0$ gives

$$\begin{aligned}
h[0] &= 0.15h[-2] + 1 = 1 \\
h[1] &= 0.15h[-1] + 0 = 0 \\
h[2] &= 0.15h[0] + 0 = 0.15 \\
h[3] &= 0.15h[1] + 0 = 0 \\
h[4] &= 0.15h[2] + 0 = 0.15^2 \\
h[5] &= 0.15h[3] + 0 = 0
\end{aligned}$$

or

$$h[n] = \begin{cases} 0.15^{n/2} & \text{for } n \geq 0 \text{ and even} \\ 0 & \text{otherwise} \end{cases}$$

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(b) Letting the initial conditions be zero, the input $x[n] = \delta[n]$ and $y[n] = h[n]$ we have

$$h[n] = 0.15h[n-2] + \delta[n]$$

replacing $h[n-2] = 0.15h[n-4] + \delta[n-2]$ according to the equation for the difference equation, we get

$$h[n] = 0.15(0.15h[n-4] + \delta[n-2]) + \delta[n] = 0.15^2 h[n-4] + 0.15\delta[n-2] + \delta[n]$$

and again replacing $h[n-4] = 0.15h[n-6] + \delta[n-4]$ we have

$$\begin{aligned} h[n] &= 0.15^2(0.15h[n-6] + \delta[n-4]) + 0.15\delta[n-2] + \delta[n] \\ &= 0.15^3 h[n-6] + 0.15^2\delta[n-4] + 0.15\delta[n-2] + \delta[n] \end{aligned}$$

Repeating this process we realize that after more iterations we get that,

$$h[n] = \delta[n] + 0.15\delta[n-2] + 0.15^2\delta[n-4] + 0.15^3\delta[n-6] + \cdots$$

which is the same result as in the previous part.

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Exercise 7

FIR filter—MATLAB

An FIR filter has a nonrecursive input–output relation

$$y[n] = \sum_{k=0}^{5} kx[n-k]$$

(a) Find and plot using MATLAB the impulse response $h[n]$ of this filter.

(b) Is this a causal and stable filter? Explain.

(c) Find and plot the unit-step response $s[n]$ for this filter.

(d) If the input $x[n]$ for this filter is bounded, i.e., $|x[n]| < 3$, what would be a minimum bound M for the output (i.e., $|y[n]| \leq M$)?

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answer

Pr 8.18 (a) Letting $x[n] = \delta[n]$ the impulse response of the filter is

$$\begin{aligned} h[n] &= 0\delta[n] + \delta[n-1] + 2\delta[n-2] + 3\delta[n-3] + 4\delta[n-4] + 5\delta[n-5] \\ &= \sum_{k=1}^{5} k\delta[n-k] \end{aligned}$$

(b) The filter is causal since the output depends only on previous values of the input, and besides $h[n] = 0$ for $n < 0$. The system is BIBO stable since the sum of the values of $|h[n]|$ is finite.

(c) Letting the input be $x[n] = u[n]$ then the unit-step response is

$$s[n] = \sum_{k=1}^{5} ku[n-k] = u[n-1] + 2u[n-2] + 3u[n-3] + 4u[n-4] + 5u[n-5]$$

The values of $s[n]$ increase from 0 to 15 (sum of the amplitudes of the delayed unit-step signals) and remain there (steady state)

(d) If the input $x[n]$ for the filter is bounded so that $|x[n]| < 3$, for the output

$$|y[n]| \leq \sum_{k=0}^{5} k|x[k]| < 3\sum_{k=0}^{5} k = 3 \times 15 = 45$$

i.e., the bound $M = 45$.

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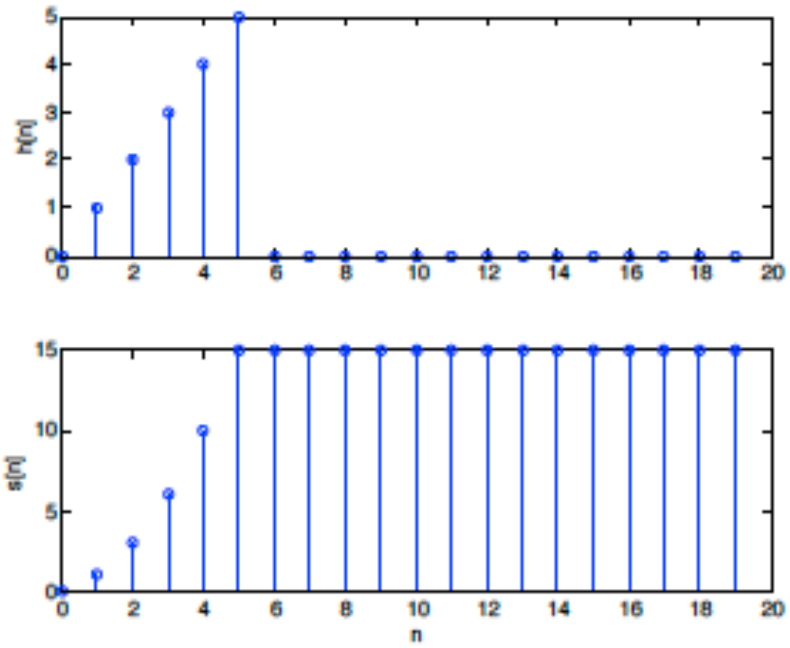
answer

impulse response

step response



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Exercise 8

Steady state of IIR systems—MATLAB

Suppose an IIR system is represented by a difference equation

$$y[n] = ay[n-1] + x[n]$$

where $x[n]$ is the input and $y[n]$ is the output.

(a) If the input $x[n] = u[n]$ and it is known that the steady-state response is $y[n] = 2$, what would be a for that to be possible (in steady state $x[n] = 1$ and $y[n] = y[n-1] = 2$ since $n \to \infty$).

(b) Writing the system input as $x[n] = u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots$ then according to the linearity and time invariance, the output should be

$$y[n] = h[n] + h[n-1] + h[n-2] + \cdots$$

Use the value for a found above, that the initial condition is zero (i.e., $y[-1] = 0$) and that the input is $x[n] = u[n]$, to find the values of the impulse response $h[n]$ for $n \geq 0$ using the above equation. The system is causal.

answer

Pr 8.20 (a) If the steady state exists then $y[n] = y[n-1] = 2$ as $n \to \infty$, and since $x[n] = 1$ for $n \geq 1$, the value of a should be according to the difference equation

$$2 = 2a + 1 \quad \text{or} \quad a = 0.5$$

(b) Using causality, $h[n] = 0$ for $n < 0$, and that the input is $x[n] = u[n]$, the initial condition is zero, $y[-1] = 0$ and $a = 0.5$, as found above, we have according to the result from linearity and time-invariance and the difference equation:

$$\begin{aligned} y[0] &= h[0] \\ &= 0.5y[-1] + 1 = 1 \\ y[1] &= h[1] + h[0] \\ &= 0.5y[0] + 1 = 0.5 + 1 \\ y[2] &= h[2] + h[1] + h[0] \\ &= 0.5y[1] + 1 = 0.5(0.5 + 1) + 1 \\ &\cdots \end{aligned}$$

solving for the impulse response values from the above equations gives:

$$\begin{aligned} h[0] &= 1 \\ h[1] &= (1 + .5) - h[0] = 0.5 \\ h[2] &= 1 + 0.5 + 0.5^2 - h[1] - h[0] = 0.5^2 \\ &\vdots \end{aligned}$$

indicating that the impulse response is $h[n] = 0.5^n u[n]$.

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Exercise 9

IIR versus FIR systems

A significant difference between IIR and FIR discrete-time systems is stability. Consider an IIR filter with the difference equation

$$y_1[n] = x[n] - 0.5y_1[n-1]$$

where $x[n]$ is the input and $y_1[n]$ is the output. Then consider an FIR filter

$$y_2[n] = x[n] + 0.5x[n-1] + 3x[n-2] + x[n-5]$$

where $x[n]$ is the input and $y_2[n]$ is the output.

(a) Since to check the stability of these filters we need their impulse responses, find the impulse responses $h_1[n]$ corresponding to the IIR filter by recursion, and $h_2[n]$ corresponding to the FIR filter.

(b) Use the impulse response $h_1[n]$ to check the stability of the IIR filter.

(c) Use the impulse response $h_2[n]$ to check the stability of the FIR filter.

(d) Since the impulse response of a FIR filter has a finite number of nonzero terms, would it be correct to say that FIR filters are always stable? Explain.

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<u>Pr 8.22</u> (a) The impulse response of the IIR filter is found by solving the difference equation

$$h_1[n] = \delta[n] - 0.5h_1[n-1] \qquad n \geq 0$$

obtained by letting $y_1[n] = h_1[n]$, $x[n] = \delta[n]$ and initial conditions equal to zero. Recursively we obtain

$$\begin{aligned} h_1[0] &= 1 \\ h_1[1] &= -0.5h_1[0] = -0.5 \\ h_1[2] &= -0.5h_1[1] = (-0.5)^2 \\ &\cdots \end{aligned}$$

which gives in general $h_1[n] = (-0.5)^n u[n]$.
The impulse response of the FIR system is obtained by letting $x[n] = \delta[n]$ and $y_2[n] = h_2[n]$ so that

$$h_2[n] = \delta[n] + 0.5\delta[n-1] + 3\delta[n-2] + \delta[n-5]$$

(b)(c) The condition for BIBO stability of a LTI system is that its impulse response be absolutely summable. The given IIR and FIR systems are LTI. The stability condition for the IIR system is that the following sum converges

$$\sum_{n=0}^{\infty} |h_1[n]| < \infty$$

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It can be shown that this sum converges, i.e.,

$$\sum_{n=0}^{\infty} |(-0.5)^n| = \sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2$$

therefore the IIR system is BIBO stable.
For the FIR, the stability condition is satisfied since we are adding a finite number of values. Indeed,

$$\sum_{n=0}^{\infty} |h_2[n]| = 1 + 0.5 + 3 + 1 < \infty$$

thus the FIR is BIBO stable.
(d) Because of the finite support of the impulse response of an FIR filter it is always absolutely summable and therefore FIR filters are always BIBO stable.

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Exercise 10

Unit-step versus impulse response—MATLAB
The unit-step response of a discrete-time LTI system is

$$s[n] = 2[(-0.5)^n - 1]u[n]$$

Use this information to find

(a) The impulse response $h[n]$ of the discrete-time LTI system.
(b) The response of the LTI system to a ramp signal $x[n] = nu[n]$.

answer

Pr 8.23 (a) For a LTI system, if $s[n]$ is the response to $u[n]$, then the response to $\delta[n] = u[n] - u[n-1]$ is $h[n] = s[n] - s[n-1]$, i.e., the impulse response of the system. So we have that

$$\begin{aligned} h[n] &= s[n] - s[n-1] = (2(0.5)^n - 2)u[n] - (2(0.5)^{n-1} - 2)u[n-1] \\ &= \begin{cases} 0 & n \leq 0 \\ -2(0.5)^n & n \geq 1 \end{cases} \end{aligned}$$

(b) The ramp in terms of the unit-step signal is given as

$$r[n] = \sum_{k=0}^{\infty} u[n-k]$$

so that the response of the system to a ramp is

$$\begin{aligned} \rho[n] &= \sum_{k=0}^{\infty} s[n-k] \\ &= \sum_{k=0}^{\infty} 2[(-0.5)^{n-k} - 1]u[n-k] \end{aligned}$$

which tends to infinity as $n \to \infty$. A close form of the ramp response is difficult to obtain, so we use MATLAB to compute it at each sample and to plot it.

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Exercise 11

Convolution sum—MATLAB

A discrete-time system has a unit-impulse response $h[n]$.

(a) Let the input to the discrete-time system be a pulse $x[n] = u[n] - u[n-4]$. Compute the output of the system in terms of the impulse response.

(b) Let $h[n] = 0.5^n u[n]$. What would be the response of the system $y[n]$ to $x[n] = u[n] - u[n-4]$? Plot the output $y[n]$.

(c) Use the convolution sum to verify your response $y[n]$.

answer

<u>Pr 8.24</u> (a) The input can be written as

$$x[n] = \sum_{k=0}^{3} \delta[n-k]$$

so the system response is

$$y[n] = \sum_{k=0}^{3} h[n-k]$$

(b) If $h[n] = 0.5^n u[n]$ then

$$\begin{aligned}
y[n] &= \sum_{k=0}^{3} 0.5^{n-k} u[n-k] = \sum_{m=n-3}^{n} 0.5^m u[m] \\
&= 0.5^n u[n] + 0.5^{n-1} u[n-1] + 0.5^{n-2} u[n-2] + 0.5^{n-3} u[n-3] \\
&= 0.5^n (u[n] + 2u[n-1] + 2^2 u[n-2] + 2^3 u[n-3]) \\
&= \begin{cases} 1 & n=0 \\ 0.5(1+2) = 3/2 & n=1 \\ 0.25(1+2+4) = 7/4 & n=2 \\ 0.5^n(1+2+4+8) = 15 \times 0.5^n & n \geq 3 \end{cases}
\end{aligned}$$

(c) Consider the input to be $x_1[n] = u[n]$, the convolution sum would be for $n \geq 0$

$$\begin{aligned} y_1[n] &= \sum_{k=-\infty}^{\infty} h[n-k]x_1[k] = 0.5^n \sum_{k=-\infty}^{\infty} 0.5^{-k}u[n-k]u[k] \\ &= 0.5^n \sum_{k=0}^{n} 2^k = 0.5^n \frac{1-2^{(n+1)}}{1-2} = 0.5^n(2^{n+1}-1) \end{aligned}$$

and zero otherwise. So that

$$y_1[n] = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ 0.5 \times 3 = 3/2 & n = 1 \\ 0.5^2 \times 7 = 7/4 & n = 2 \\ 0.5^3 \times 15 = 15/8 & n = 3 \\ \vdots & \vdots \end{cases}$$

then the output due to $x[n] = u[n] - u[n-4]$ is by superposition

$$\begin{aligned} y[n] &= 0.5^n(2^{n+1}-1)u[n] - 0.5^{n-4}(2^{n+1-4}-1)u[n-4] \\ &= \begin{cases} 0 & n < 0 \\ 0.5^n(2^{n+1}-1) & 0 \leq n \leq 3 \\ 0.5^n(2^{n+1}-1) - 0.5^{n-4}(2^{n-3}-1) = 0.5^n(15) & n \geq 4 \end{cases} \end{aligned}$$

which coincides with the result obtained above.