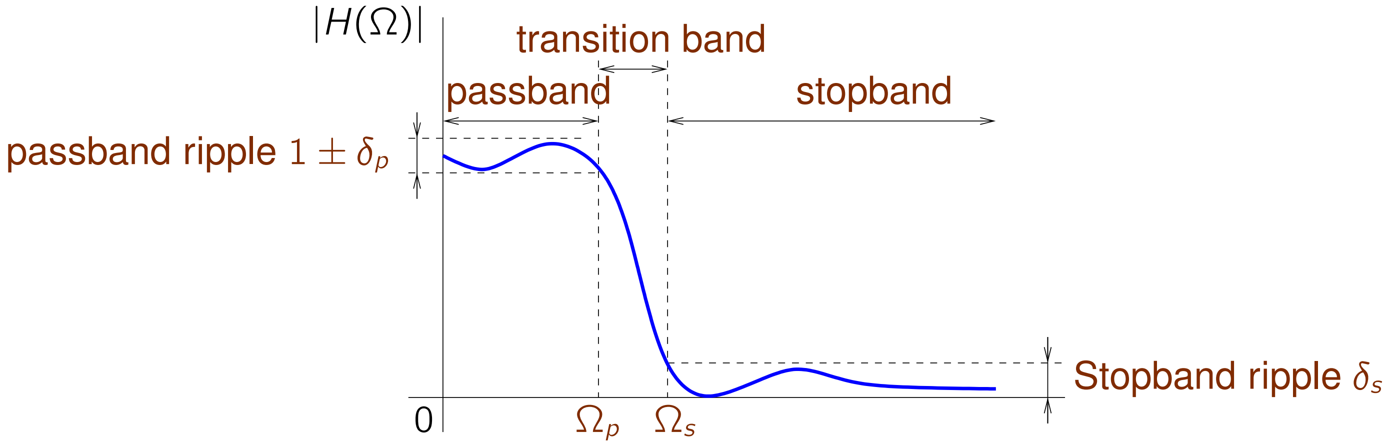
Ch.7.3 Analog filter design

How can I design an analog filter $H(s)$ that meets certain specifications?



Note differences in notation. We often write $H(\Omega)$ instead of $H(j\Omega)$.

Analog filter design

Continuous-time filter functions

- General form:

$$H(s) = \frac{B(s)}{A(s)} = \frac{b_0 + b_1 s + \cdots + b_n s^n}{1 + a_1 s + \cdots + a_n s^n}$$

- Stability and causality:

$$\begin{aligned} & \text{poles of } H(s) \text{ in left half plane} \\ \Leftrightarrow \quad & \text{zeros of } A(s) \text{ in left half plane} \end{aligned}$$

- Frequency spectrum: $|H(\Omega)|^2 = |H(j\Omega)|^2 = H(s)H(-s)\Big|_{s=j\Omega}$

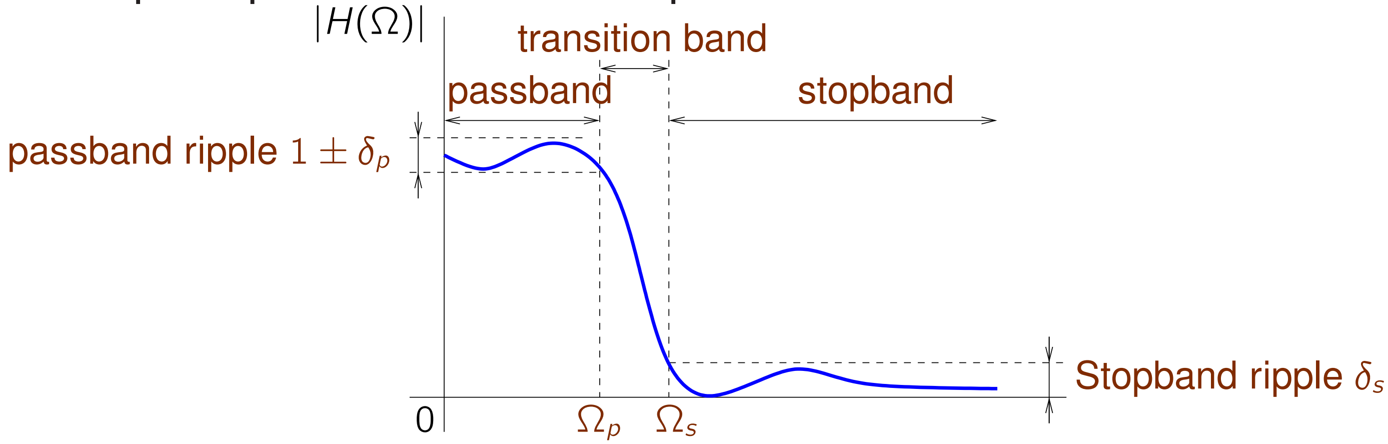
  Damping (loss): $\alpha(\Omega) = \dfrac{1}{|H(\Omega)|^2}$;

  usually specified in dB: $\alpha(\Omega)[\text{dB}] = -10\log(|H(\Omega)|^2) = -20\log(|H(\Omega)|)$

Analog filter design

Filter specifications

- Example: specification of a low-pass filter



- In the same manner: high-pass, band-pass, band-stop

- An ideal filter does not have ripples and no transition band. But, a causal filter has a finite number of zeros and cannot be an ideal filter (Paley-Wiener).

  $|H(\Omega)|$ cannot be constant over an interval.

- Usually, only the amplitude spectrum is specified, because the phase spectrum is (almost) completely determined by this (cf. the *Hilbert transform* or the causality requirement)

Analog filter design

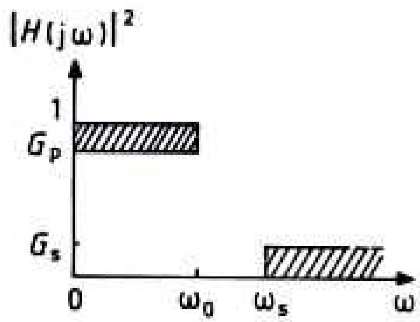
Practical design

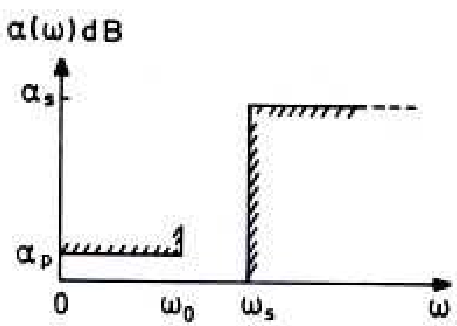
We limit ourselves to design techniques based on amplitude specifications.

We start with low-pass filters (the other types are derived from these).

Specifications: like before,

- Ω_p also written as Ω_0: pass-band frequency
- G_p: minimal squared-amplitude in the pass-band, or α_p maximal damping
- Ω_s: stop-band frequency
- G_s: maximal squared-amplitude (α_s minimal damping) in the stop-band

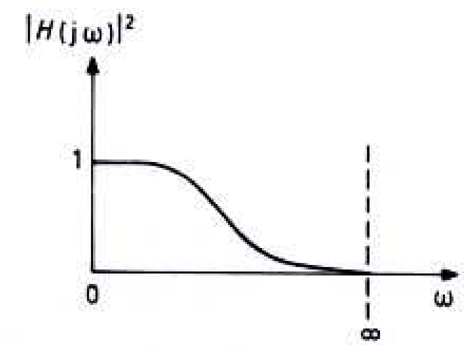




Butterworth filter

- We start from the following characteristics

  - $|H(j\Omega)|^2$ is an even function
  - $H(s)$ is rational, order n
  - $\lim_{\Omega\to 0} |H(j\Omega)|^2 = 1$
  - $\lim_{\Omega\to\infty} |H(j\Omega)|^2 = 0$



so that

$$|H(j\Omega)|^2 = \frac{1 + \sum_{r=1}^{n-1} b_r \Omega^{2r}}{1 + \sum_{r=1}^{n} a_r \Omega^{2r}}$$

- The "Butterworth filter" is obtained if we require $|H(j\Omega)|^2$ to be maximally flat for $\Omega = 0$ and $\Omega = \infty$:

$2n-1$ derivatives equal to zero in $\Omega = 0 \quad \Rightarrow \quad a_r = b_r \qquad r = 1, 2, \dots, n-1$

$2n-1$ derivatives equal to zero in $\Omega = \infty \quad \Rightarrow \quad b_r = 0 \qquad r = 1, 2, \dots, n-1$

$$|H(j\Omega)|^2 = \frac{1}{1 + a_n \Omega^{2n}} \qquad \text{or} \qquad |H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 \Omega^{2n}}$$

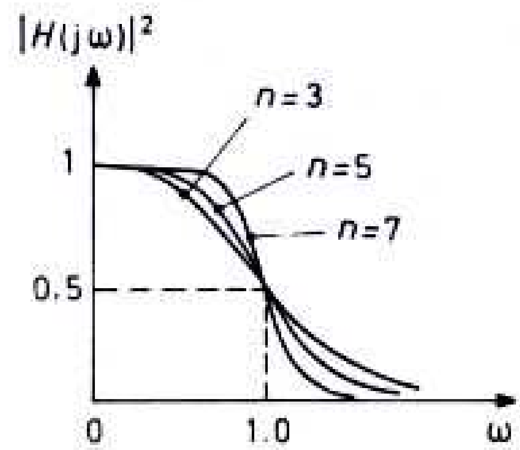
Analog filter design

Butterworth filter

Filter parameters are ϵ and n. How to design them?

- Example ($\epsilon = 1$):

$$|H(j\Omega)|^2 = \frac{1}{1+(\Omega)^{2n}}$$



- Larger $n \Rightarrow$ steeper roll-off (smaller transition band)
- Independent of n, these filters have a cutoff frequency (3 dB damping) at $\Omega_c = 1$:

$$|H(\Omega_c = 1)|^2 = \frac{1}{1+(1)^{2n}} = \frac{1}{2} \qquad \Rightarrow \qquad \alpha(\Omega_c) = -10\log\left(\frac{1}{2}\right) = 3 \text{ dB}$$

Analog filter design

Butterworth filter

- What if we want a 3 dB point at some other Ω_c? Use as template

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2n}}$$

- What if Ω_0 is specified, and a corresponding damping $\alpha(\Omega_0)$? Use

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2(\Omega/\Omega_0)^{2n}}$$

For this template, independent of n, we have at Ω_0

$$|H(j\Omega_0)|^2 = \frac{1}{1+\epsilon^2} \qquad \Rightarrow \qquad \alpha_p = \alpha(\Omega_0) = 10\log(1+\epsilon^2) \qquad \Rightarrow \qquad \epsilon = \sqrt{10^{\alpha_p/10} - 1}$$

Use this to determine ϵ.

Next, find the minimal n from the damping condition at Ω_s.

Analog filter design

Example 1: Design Butterworth filter

- Determine the minimal order of the Butterworth filter with pass-band frequency $F_0 = 1.2$ kHz, maximal damping in the pass-band $\alpha_p = 0.5$ dB, stop-band frequency $F_s = 1.92$ kHz, and minimal damping in the stop-band $\alpha_s = 23$ dB

- Solution:

  - We start from

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega/\Omega_0)^{2n}} \qquad \text{with} \qquad \Omega_s/\Omega_0 = F_s/F_0 = 1.6$$

  - From $\alpha_p = \alpha(\Omega_0)$ we derive ϵ:

$$|H(\Omega_0)|^2 = \frac{1}{1+\epsilon^2} = 10^{-\alpha_p/10} \quad \Rightarrow \quad \epsilon = \sqrt{10^{\alpha_p/10}-1} = 0.3493$$

  - From ϵ, Ω_s/Ω_0 and α_s we derive the minimal n:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\Omega_s/\Omega_0)^{2n}} = 10^{-\alpha_s/10} \quad \Rightarrow \quad n \geq \frac{\log[(10^{\alpha_s/10}-1)/\epsilon^2]}{2\log(\Omega_s/\Omega_0)} = 7.87$$

Analog filter design

What is $H(s)$ for the Butterworth filter?

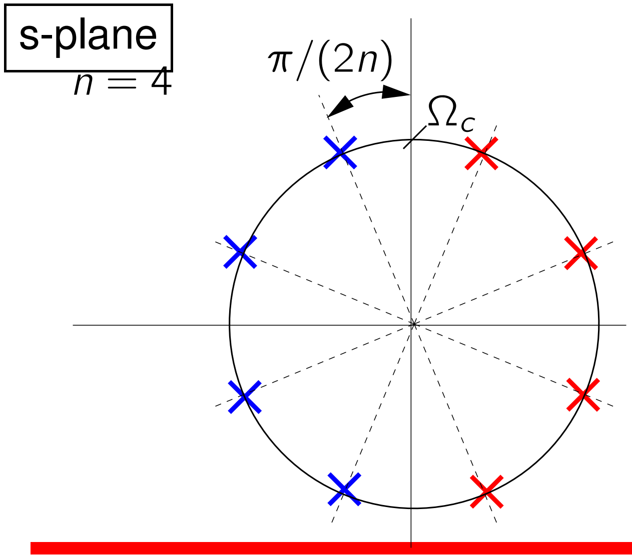
- From $|H(s)|^2\Big|_{s=j\Omega} = |H(j\Omega)|^2 = H(j\Omega)H(-j\Omega) = H(s)H(-s)\Big|_{s=j\Omega}$ it follows

$$H(s)H(-s) = |H(j\Omega)|^2\Big|_{\Omega=-js} = \frac{1}{1+\epsilon^2(-js)^{2n}} = \frac{1}{1+\epsilon^2(-s^2)^n}$$

- The poles of $H(s)H(-s)$ follow as

$$(-js_k)^{2n} = -1/\epsilon^2 \quad \Rightarrow \quad s_k = (1/\epsilon)^{1/n} e^{j[(2k-1)\pi/(2n)+\pi/2]}, \quad k = 1, 2, \ldots, 2n$$

These are located on a circle with radius $(1/\epsilon)^{1/n} = \Omega_c$



Stable: poles of $H(s)$ are the n values in the left-half plane

Analog filter design

Chebyshev filter

The Butterworth filter has maximal error in the pass-band at Ω_0, elsewhere the error is smaller. Perhaps the filter order can be made smaller (or the response sharper for the same filter order) by distributing the error more uniformly over the pass-band?

- We keep the maximal flatness in $\Omega = \infty$:

$$2n-1 \text{ derivatives zero in } \Omega = \infty \quad \Rightarrow \quad b_r = 0, \qquad r = 1, 2, \ldots, n-1$$

$$|H(j\Omega)|^2 = \frac{1}{1 + \sum_{r=1}^{n} a_r \Omega^{2r}}$$

- In the case of Chebyshev this is written more specifically as

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)}$$

$T_n(\Omega)$ is an even or odd polynomial of order n (because $T_n^2(\Omega)$ has to be even)

- In the pass-band we must have: $|T_n(\Omega)| \le 1$. Elsewhere: $|T_n(\Omega)| \to \infty$

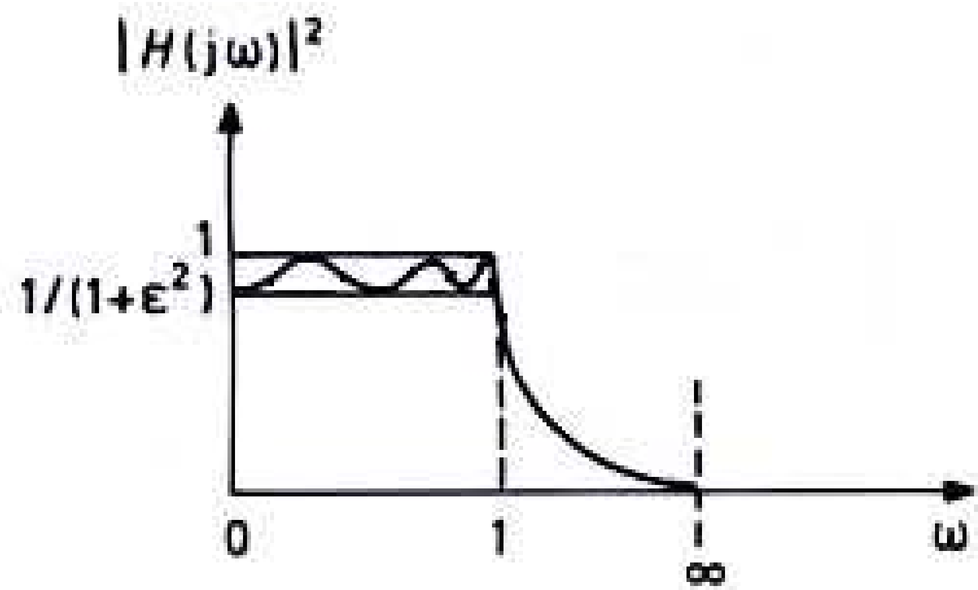
Chebyshev filter

From now on, normalize the pass-band to $-1 \leq \Omega \leq 1$:

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2(\Omega)}$$

with

$$|T_n(\Omega)| \leq 1 \quad (|\Omega| < 1)$$
$$|T_n(\Omega)| \to \infty \quad (|\Omega| \to \infty)$$



Example polynomials:

$$\begin{aligned} T_0(\Omega) &= 1 \\ T_1(\Omega) &= \Omega \\ T_2(\Omega) &= 2\Omega^2 - 1 \\ T_3(\Omega) &= 4\Omega^3 - 3\Omega \end{aligned}$$

Analog filter design

Chebyshev polynomials

Idea: $T_n(\Omega)$ has to oscillate between -1 and 1 in the pass-band, hence set

$$T_n(\Omega) = \cos(n\,\theta(\Omega)) \qquad -1 \leq \Omega \leq 1$$

How do we design $\theta(\Omega)$ such that $T_n(\Omega)$ is an even or odd polynomial of order n?

- From the property $\cos(\alpha+\beta)+\cos(\alpha-\beta) = 2\cos\alpha\cos\beta$ we obtain the recursion

$$T_n(\Omega) = 2T_{n-1}(\Omega)\cos(\theta(\Omega)) - T_{n-2}(\Omega) \quad \text{with } T_0(\Omega) = 1 \text{ and } T_1(\Omega) = \cos(\theta(\Omega))$$

- Repeat this to obtain

$$T_n(\Omega) = c_n[\cos(\theta(\Omega))]^n + c_{n-2}[\cos(\theta(\Omega))]^{n-2} + c_{n-4}[\cos(\theta(\Omega))]^{n-4} + \cdots$$

- $T_n(\Omega)$ is an even or odd polynomial in Ω of order n if $\theta(\Omega) = \cos^{-1}\Omega$ ($|\Omega| \leq 1$).

  This gives $\cos(\theta(\Omega)) = \Omega$.

- Also valid: $\cosh(\alpha+\beta) + \cosh(\alpha-\beta) = 2\cosh\alpha\cosh\beta$ $\rightarrow$ same recursion!

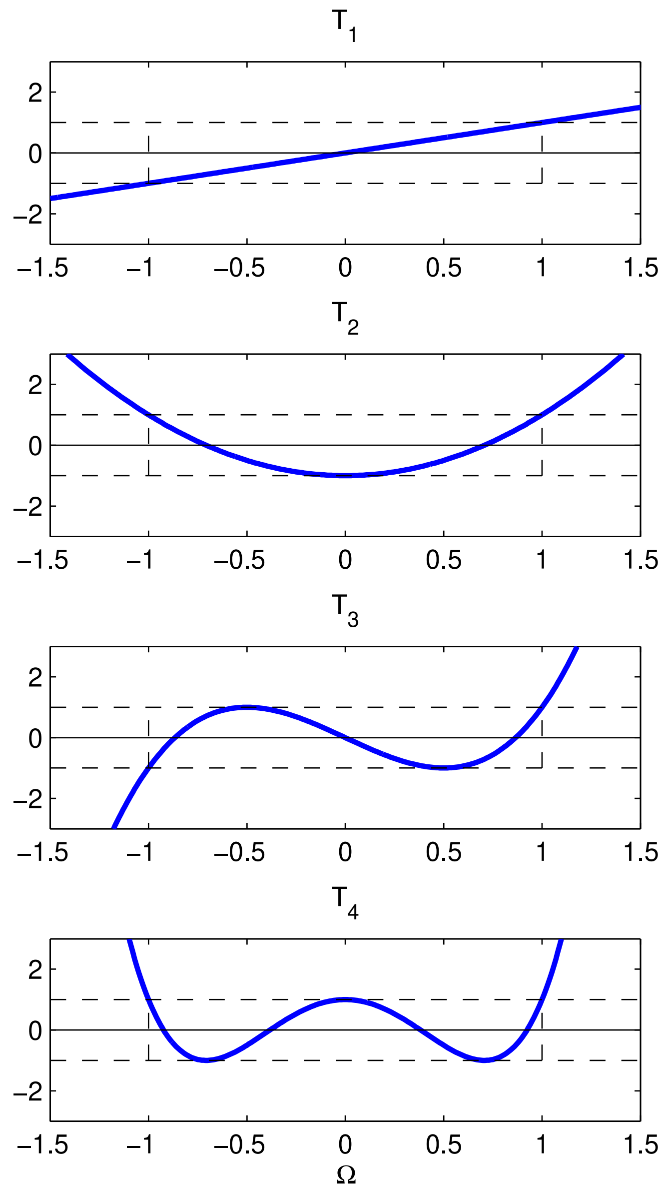
Chebyshev polynomials

The recursion becomes:

$T_n(\Omega) = 2\Omega T_{n-1}(\Omega) - T_{n-2}(\Omega)$:

$$
\begin{aligned}
T_0(\Omega) &= 1 \\
T_1(\Omega) &= \Omega \\
T_2(\Omega) &= 2\Omega^2 - 1 \\
T_3(\Omega) &= 4\Omega^3 - 3\Omega \\
T_4(\Omega) &= 8\Omega^4 - 8\Omega^2 + 1 \\
T_5(\Omega) &= 16\Omega^5 - 20\Omega^3 + 5\Omega \\
&\vdots
\end{aligned}
$$

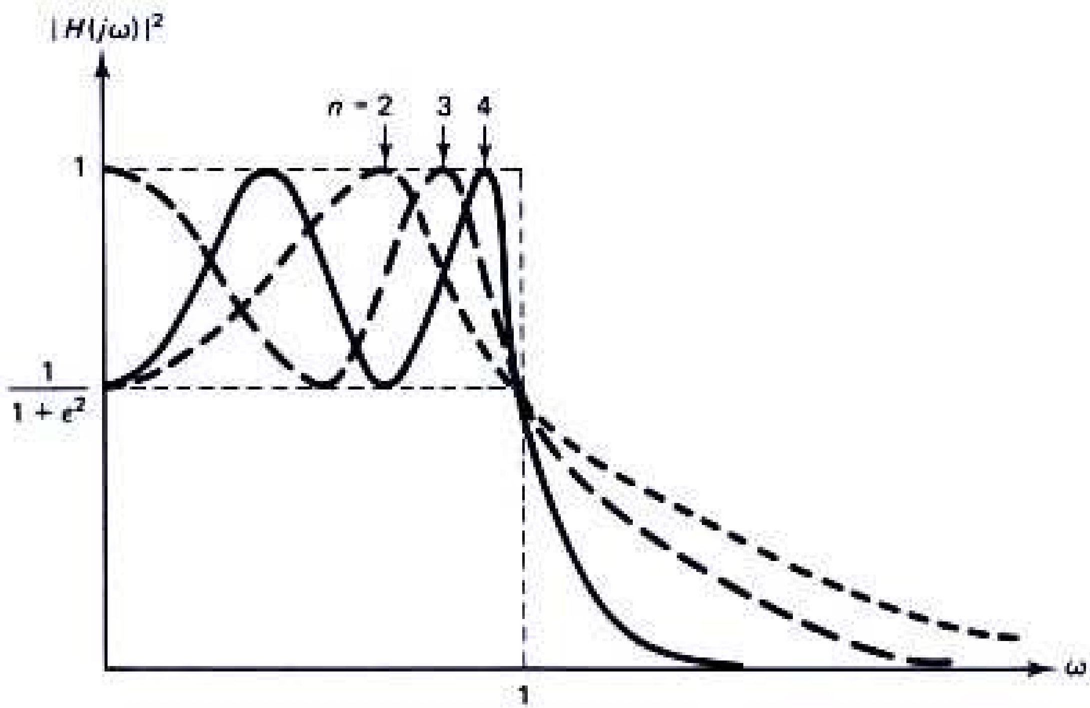
$$
T_n(\Omega) = \begin{cases} \cos(n\cos^{-1}(\Omega)), & (|\Omega| \le 1) \\ \cosh(n\cosh^{-1}(\Omega)), & (|\Omega| > 1) \end{cases}
$$



Analog filter design

- Resulting filters:

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2(\Omega)}\,, \qquad \text{or more general} \quad |H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2\left(\frac{\Omega}{\Omega_0}\right)}$$



Analog filter design

Design of Chebyshev filters

For the design of ϵ and n, we usually start from

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2\left(\frac{\Omega}{\Omega_0}\right)}$$

with

$$T_n(\Omega) = \begin{cases} \cos(n\cos^{-1}(\Omega)), & (|\Omega| \leq 1) \\ \cosh(n\cosh^{-1}(\Omega)), & (|\Omega| > 1) \end{cases}$$

- If Ω_0 and a damping $\alpha(\Omega_0)$ is specified: [using that $T_n^2(1) = 1$ for any n]

  $$|H(j\Omega_0)|^2 = \frac{1}{1+\epsilon^2} \qquad \Rightarrow \qquad \alpha_p = \alpha(\Omega_0) = 10\log(1+\epsilon^2) \qquad \Rightarrow \qquad \epsilon = \sqrt{10^{\alpha_p/10} - 1}$$

  Same as for Butterworth! Use this to determine ϵ from the specs.

- Next, find n from the damping condition at Ω_s. You will need to evaluate $T_n(\Omega_s/\Omega_0)$.

  Since $\Omega_s > \Omega_0$, use the "cosh" formula.

Analog filter design

Example 2: Design Chebyshev filter

- Determine the minimal order of a Chebyshev filter with pass-band frequency $F_0 = 1.2$ kHz, maximal damping in the pass-band $\alpha_p = 0.5$ dB, stop-band frequency $F_s = 1.92$ kHz, and minimal damping in the stop-band $\alpha_s = 23$ dB

- Solution:

  - We start from

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 T_n^2(\Omega/\Omega_0)} \qquad \text{with} \qquad \Omega_s/\Omega_0 = F_s/F_0 = 1.6$$

  - From $\alpha_p = \alpha(\Omega_0)$ we derive ϵ:

$$|H(\Omega_0)|^2 = \frac{1}{1+\epsilon^2} = 10^{-\alpha_p/10} \quad \Rightarrow \quad \epsilon = \sqrt{10^{\alpha_p/10} - 1} = 0.3493$$

  - From ϵ, Ω_s/Ω_0 and α_s we derive the minimal order n:

$$|H(\Omega_s)|^2 = \frac{1}{1+\epsilon^2[\cosh(n\cosh^{-1}(\Omega_s/\Omega_0))]^2} = 10^{-\alpha_s/10} \quad \Rightarrow \quad n \geq \frac{\cosh^{-1}(\sqrt{(10^{\alpha_s/10}-1)/\epsilon^2})}{\cosh^{-1}(\Omega_s/\Omega_0)} = 3.82$$

$\cosh(x) = \frac{1}{2}(e^x + e^{-x}) \Rightarrow \cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$

What is $H(s)$ for the Chebyshev filter?

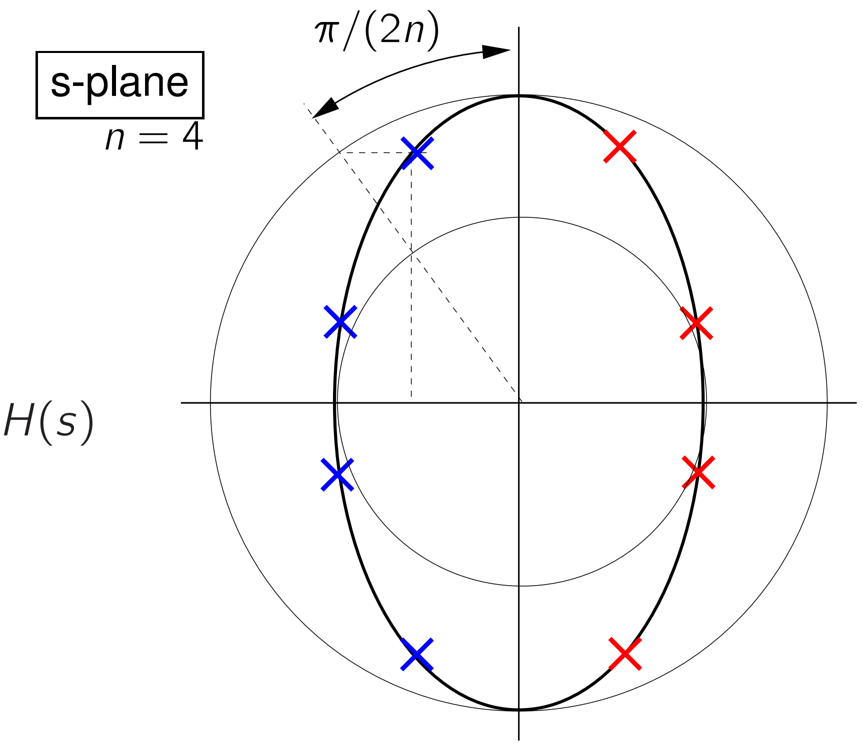
- Like with the Butterworth filter we look for $H(s)$ for which

$$H(s)H(-s) = |H(j\Omega)|^2\Big|_{\Omega=-js} = \frac{1}{1+\epsilon^2 T_n^2(-js)}$$

- Poles of $H(s)H(-s)$ satisfy

$$\begin{aligned} T_n^2(-js_k) &= -1/\epsilon^2 \\ \Rightarrow \quad s_k &= \sigma_k + j\Omega_k, \quad k = 1, \cdots, 2n \end{aligned}$$

These turn out to lie on an ellipse. Poles of $H(s)$ are the n poles in the left-half plane

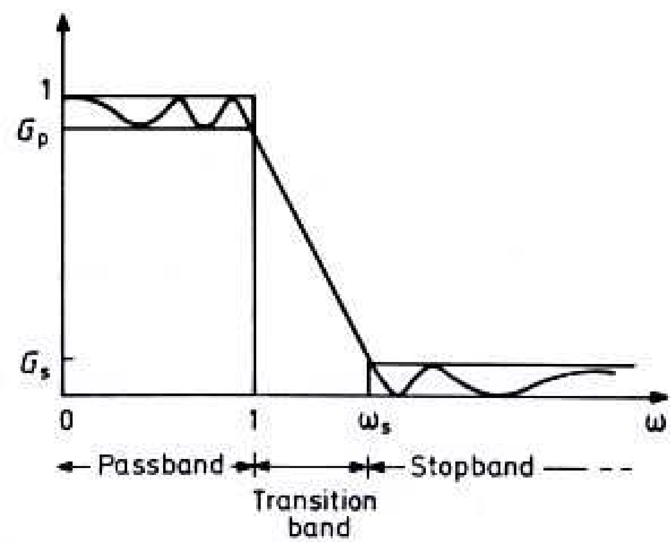


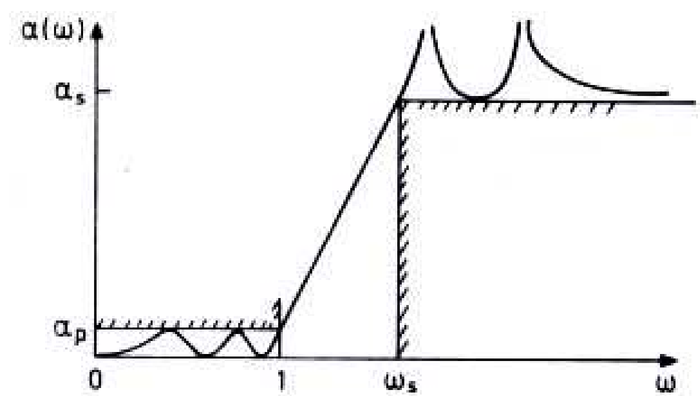
Elliptic filter

- Generalization of the Chebyshev filter:

$$|H(j\Omega)|^2 = \frac{1}{1+\epsilon^2 R_n^2(\Omega)}$$

with $R_n(\Omega)$ an arbitrary rational function in Ω





We will not discuss this any further.

Analog filter design

Frequency transformations

To transform a prototype filter into a desired filter we use transformations of the frequency axis:

- low-pass to low-pass : shift a frequency from $\Omega = 1$ to $\Omega = \Omega_0$:

  substitute: $$\Omega \rightarrow \frac{\Omega}{\Omega_0} \qquad s \rightarrow \frac{s}{\Omega_0}$$

- More generally: shift a frequency of $\Omega = \Omega_0$ to $\Omega = \Omega_0'$:

  substitute: $$\Omega \rightarrow \Omega \frac{\Omega_0}{\Omega_0'} \qquad s \rightarrow s \frac{\Omega_0}{\Omega_0'}$$

Example: low-pass to low-pass

Suppose we have a template filter with cut-off frequency $\Omega = 1$:

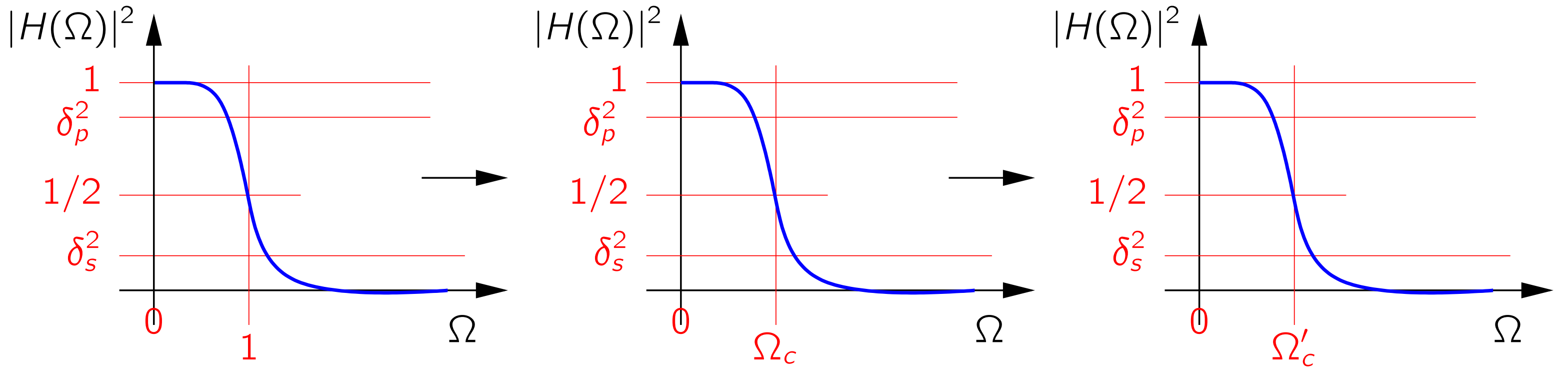
$$|H(\Omega)|^2 = \frac{1}{1+\Omega^{2n}}$$

Mapping to a filter with cut-off frequency $\Omega = \Omega_c$: transform $\Omega \to \frac{\Omega}{\Omega_c}$

$$|H(\Omega)|^2 = \frac{1}{1+(\frac{\Omega}{\Omega_c})^{2n}}$$

Mapping to a filter with cut-off frequency $\Omega = \Omega'_c$: transform $\Omega \to \frac{\Omega\Omega_c}{\Omega'_c}$

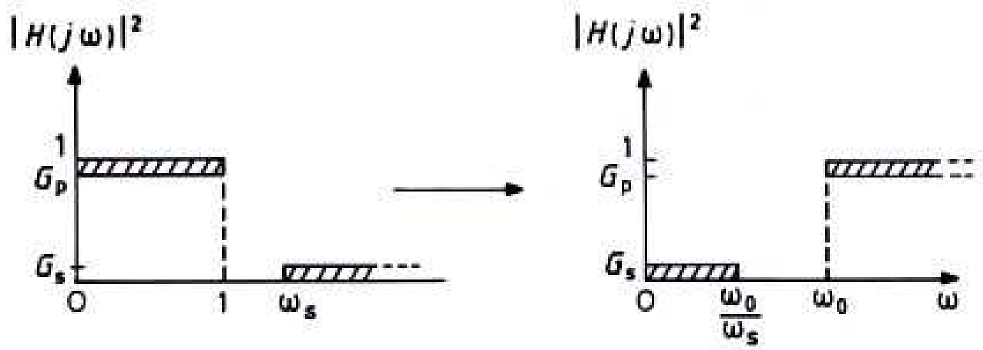
$$|H(\Omega)|^2 = \frac{1}{1+(\frac{\Omega}{\Omega'_c})^{2n}}$$



Frequency transforms (2)

- low-pass to high-pass: mapping $1 \rightarrow \Omega_0$, and $\Omega_s \rightarrow \Omega_0/\Omega_s$

$$\Omega \rightarrow \frac{\Omega_0}{\Omega} \qquad s \rightarrow \frac{\Omega_0}{s}$$



- More generally: mapping $\Omega_0 \rightarrow \Omega_0'$ with reversal of the frequency axis

$$\Omega \rightarrow \frac{\Omega_0 \Omega_0'}{\Omega}, \qquad s \rightarrow \frac{\Omega_0 \Omega_0'}{s}$$

Analog filter design

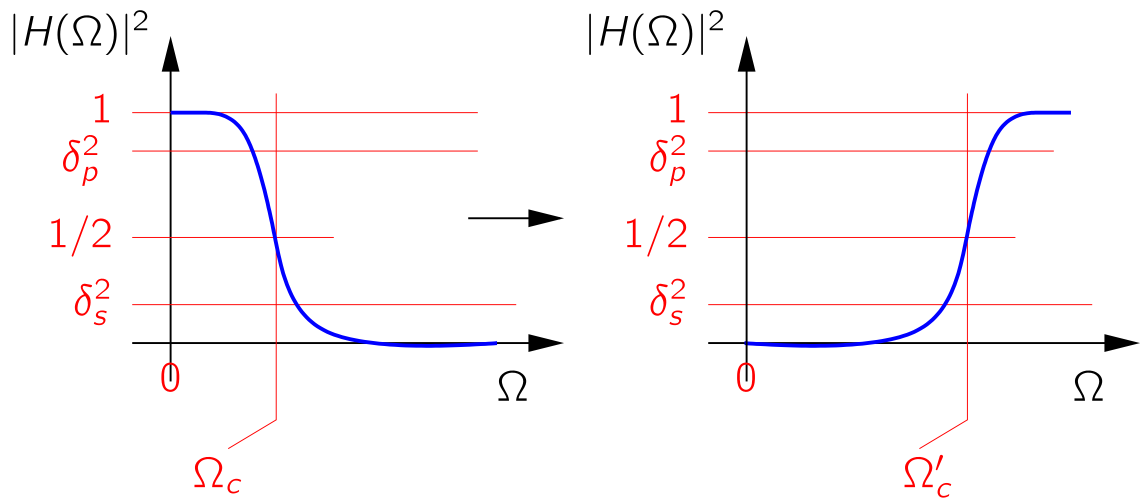
Example: low-pass to high-pass

Suppose the template low-pass filter has cut-off frequency $\Omega = \Omega_c$:

$$|H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega}{\Omega_c})^{2n}}$$

Transform $\Omega \to \frac{\Omega_c \Omega'_c}{\Omega}$ gives a high-pass filter with cut-off frequency Ω'_c:

$$|H(\Omega)|^2 = \frac{1}{1 + (\frac{\Omega'_c}{\Omega})^{2n}}$$



Analog filter design

Example 3: use of the low-to-high transform

We require an analog *high-pass* filter design with the following specifications:

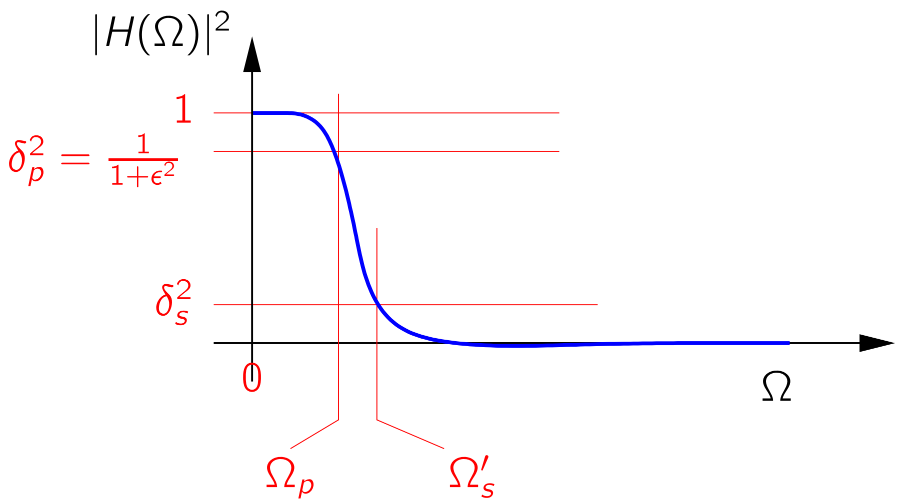
– Pass-band: starting at $F_p = 50$ Hz; ripple in the pass-band: ≤ 1 dB

– Stop-band: until $F_s = 40$ Hz; stop-band damping: ≥ 30 dB.

- We start with a Butterworth low-pass filter structure of the form

$$|H(\Omega)|^2 = \frac{1}{1 + \epsilon^2 (\Omega/\Omega_p)^{2n}}$$

which we design such that $\Omega_p = 2\pi \cdot 50$, and $|H(\Omega_p)|^2$ equal to -1 dB.

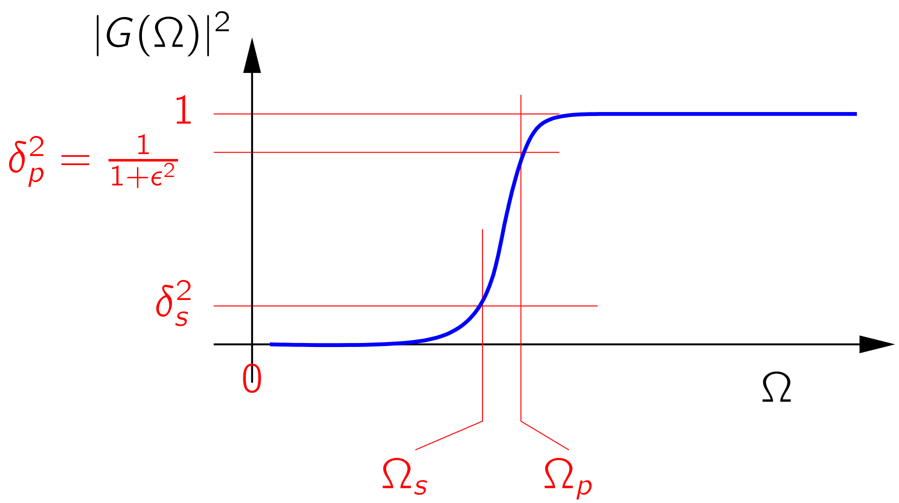


Analog filter design

- Next, we apply to $H(\Omega)$ a low-to-high transform:

$$\Omega \to \frac{\Omega_p^2}{\Omega} \qquad \text{gives} \qquad |G(\Omega)|^2 = \frac{1}{1+\epsilon^2(\Omega_p/\Omega)^{2n}}$$

This is a high-pass filter with pass-band Ω_p.



From the transformation, it follows that the stop-band frequency in the design of $H(\Omega)$ should be $\Omega_s' = \frac{\Omega_p^2}{\Omega_s} = 2\pi \cdot \frac{F_p^2}{F_s} = 2\pi \cdot 62.50$.

Instead of first designing $H(\Omega)$, we can also directly determine ϵ and n for $G(\Omega)$.

Analog filter design

So we use as template highpass filter:

$$|G(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega_p}{\Omega}\right)^{2n}}$$

- Determine ϵ by evaluation at $\Omega_p = 2\pi \cdot 50$:

$$|G(\Omega_p)|^2 = \frac{1}{1+\epsilon^2} = 10^{-1/10} \qquad \Rightarrow \qquad \epsilon = \sqrt{10^{1/10}-1} = 0.5088.$$

- Determine n by evaluation at $\Omega_s = 2\pi \cdot 40$:

$$|G(\Omega_s)|^2 = \frac{1}{1+\epsilon^2(\frac{2\pi\cdot 50}{2\pi\cdot 40})^{2n}} = 10^{-30/10} \qquad \Rightarrow \qquad (\frac{50}{40})^{2n} = \frac{10^{30/10}-1}{\epsilon^2} = 3858$$

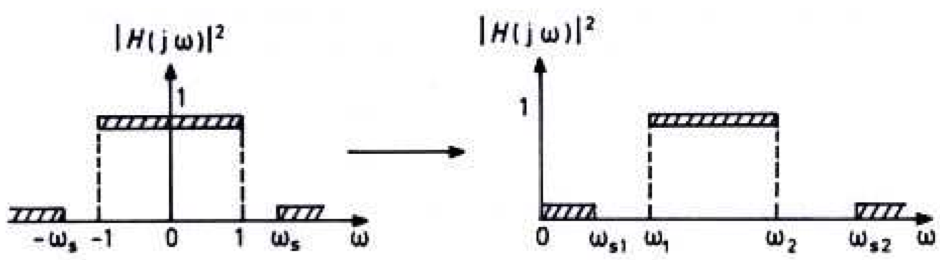
$$\Rightarrow \qquad n = \frac{1}{2}\frac{\log(3858)}{\log(5/4)} = 18.5$$

We take filter order $n = 19$.

(Advanced material: you can skip) Frequency transformations (3)

- low-pass to band-pass: Suppose the template filter has cut-off frequency 1

$$\Omega \to \beta\left(\frac{\Omega}{\Omega_0} - \frac{\Omega_0}{\Omega}\right), \qquad s \to \beta\left(\frac{s}{\Omega_0} + \frac{\Omega_0}{s}\right)$$



- This transformation should map $0 \to \Omega_0$, $-1 \to \Omega_1$, and $1 \to \Omega_2$.
- Band center Ω_0 and scale factor β are computed based on the desired cut-off frequency Ω_1 and Ω_2:

$$\left.\begin{aligned} -1 &= \beta\left(\frac{\Omega_1}{\Omega_0} - \frac{\Omega_0}{\Omega_1}\right) \\ 1 &= \beta\left(\frac{\Omega_2}{\Omega_0} - \frac{\Omega_0}{\Omega_2}\right) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} \Omega_0 &= \sqrt{\Omega_1\Omega_2} \\ \beta &= \frac{\Omega_0}{\Omega_2 - \Omega_1} \end{aligned}\right.$$

Analog filter design

- The pass-band is geometrically symmetric around Ω_0:

$$\alpha(\Omega_a) = \alpha(\Omega_b) \quad \Leftrightarrow \quad \Omega_a\Omega_b = \Omega_0^2$$

Derivation

After transformation, we must have:

$$\Omega'_a = -\Omega'_b$$

as this gives $\alpha(\Omega_a) = \alpha(\Omega_b)$. We find

$$\frac{\Omega_a}{\Omega_0} - \frac{\Omega_0}{\Omega_a} = -\left(\frac{\Omega_b}{\Omega_0} - \frac{\Omega_0}{\Omega_b}\right)$$

$$\frac{\Omega_a^2 - \Omega_0^2}{\Omega_a\Omega_0} = \frac{\Omega_0^2 - \Omega_b^2}{\Omega_0\Omega_b}$$

$$\Omega_b(\Omega_a^2 - \Omega_0^2) = \Omega_a(\Omega_0^2 - \Omega_b^2)$$

$$\Omega_0^2(\Omega_b + \Omega_a) = \Omega_b\Omega_a^2 + \Omega_a\Omega_b^2$$

$$\Omega_0^2 = \Omega_a\Omega_b$$

(Advanced material: you can skip) Frequency transformations (4)

- low-pass to band-pass (general)

  Suppose the template filter has cut-off frequency Ω_p, the new filter has cut-off frequencies Ω_l and Ω_u:

$$\Omega \to \Omega_p \frac{\Omega^2 - \Omega_l\Omega_u}{\Omega(\Omega_u - \Omega_l)}, \qquad s \to \Omega_p \frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)}$$

Verification:

– Evaluate for Ω_u: this gives $\Omega_p \frac{\Omega_u^2 - \Omega_l\Omega_u}{\Omega_u(\Omega_u - \Omega_l)} = \Omega_p$

– Evaluate for Ω_l: this gives $\Omega_p \frac{\Omega_l^2 - \Omega_l\Omega_u}{\Omega_l(\Omega_u - \Omega_l)} = -\Omega_p$

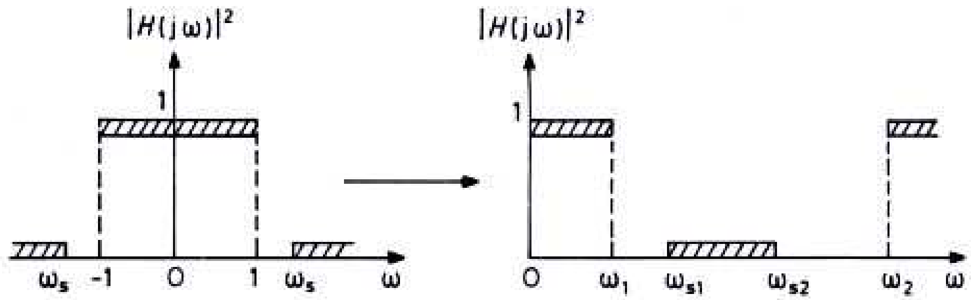
– Evaluate for $\Omega_0 = \sqrt{\Omega_l\Omega_u}$: this gives $\Omega_p \frac{\Omega_l\Omega_u - \Omega_l\Omega_u}{\sqrt{\Omega_l\Omega_u}(\Omega_u - \Omega_l)} = 0$.

Note that this transformation doubles the filter order!

Analog filter design

- low-pass to band-stop (template cut-off frequency 1)

$$\Omega \to \frac{1}{\beta\left(\frac{\Omega}{\Omega_0} - \frac{\Omega_0}{\Omega}\right)}, \qquad s \to \frac{1}{\beta\left(\frac{s}{\Omega_0} + \frac{\Omega_0}{s}\right)}$$



- Ω_0 and β are calculated based on Ω_1 and Ω_2:

$$\left.\begin{array}{r} -1 = \dfrac{1}{\beta\left(\frac{\Omega_1}{\Omega_0} - \frac{\Omega_0}{\Omega_1}\right)} \\ 1 = \dfrac{1}{\beta\left(\frac{\Omega_2}{\Omega_0} - \frac{\Omega_0}{\Omega_2}\right)} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \Omega_0 = \sqrt{\Omega_1 \Omega_2} \\ \beta = \dfrac{\Omega_0}{\Omega_2 - \Omega_1} \end{array}\right.$$

- Band-stop characteristic is geometrically symmetric around Ω_0:

$$\alpha(\Omega_a) = \alpha(\Omega_b) \quad \Leftrightarrow \quad \Omega_a \Omega_b = \Omega_0^2$$

(skip) Example 4: Frequency transformations

- Design a band-pass Chebyshev filter with pass-band $F_1 = 10$ kHz until $F_2 = 15$ kHz, maximal damping $\alpha_p = 0.28$ dB in de pass-band, and minimal damping $\alpha_s = 40$ dB for $F \leq F_{s1} = 8.5$ kHz and $F \geq F_{s2} = 17$ kHz.

- Solution: We don't have a transformation with 4 frequencies as parameters. We select a transformation based on the pass-band frequencies and then check the stop-band.

  We will use the following transformation:

$$\Omega \to \beta\left(\frac{\Omega}{\Omega_0} - \frac{\Omega_0}{\Omega}\right), \qquad s \to \beta\left(\frac{s}{\Omega_0} + \frac{\Omega_0}{s}\right)$$

  with $\Omega_0 = \sqrt{\Omega_1\Omega_2}$ and $\beta = \frac{\Omega_0}{\Omega_2 - \Omega_1}$ derived from the pass-band frequencies.

Analog filter design

Example 4 (cont'd)

- Determine the geometric center of the pass-band and the scale factor:

$$\Omega_0 = (\Omega_1 \Omega_2)^{1/2} = 2\pi \times 12.247 \times 10^3 \,, \qquad \beta = \frac{\Omega_0}{\Omega_2 - \Omega_1} = 2.4219$$

- Use the property $F_0^2 = F_1 F_2 = F_{s1} F'_{s2} = F'_{s1} F_{s2}$ to determine which side gives the strongest damping requirements in the stop-band:

$$\alpha \geq 40 \text{ dB}, \; F \leq F_{s1} = 8.5 \text{ kHz} \quad \Rightarrow \quad \alpha \geq 40 \text{ dB}, \; F \geq 10 \cdot 15/8.5 = 17.647 \text{ kHz} = F'_{s2}$$

$$\alpha \geq 40 \text{ dB}, \; F \geq F_{s2} = 17 \text{ kHz} \quad \Rightarrow \quad \alpha \geq 40 \text{ dB}, \; F \leq 10 \cdot 15/17 = 8.824 \text{ kHz} = F'_{s1}$$

- Therefore we calculate our low-pass characteristic based on $F_{s2} = 17$ kHz: if we meet 40 dB damping here, then we will also have this at $F'_{s1} = 8.824$ kHz and certainly at $F_{s1} = 8.5$ kHz.

Analog filter design

Example 4 (cont'd)

- The transform gives

$$\Omega_s = \beta\left(\frac{\Omega_{s2}}{\Omega_0} - \frac{\Omega_0}{\Omega_{s2}}\right) = 2.4219\left(\frac{17}{12.247} - \frac{12.247}{17}\right) = 1.635$$

- Thus, we have to design a template low-pass Chebyshev filter with at $\Omega_0 = 1$ a damping of 0.28 dB, and at $\Omega_s = 1.635$ a damping of 40 dB.

  Like before, we find $\epsilon = 0.258$ and $n \geq 6.19$

- Next, determine $H(s)$ and insert the transformation to obtain the desired filter.