Exercises Ch.11 DTFT

Exercise 1 – taken from Chaparro (first edition)

Computations from definition of DTFT and IDTFT

Consider the discrete-time signal $x[n] = 0.5^{|n|}$, and find its DTFT $X(e^{j\omega})$. From the direct and the inverse DTFT of $x[n]$:

(a) Determine the infinite sum

$$\sum_{k=-\infty}^{\infty} 0.5^{|n|}$$

(b) Find the integral

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$$

(c) Find the phase of $X(e^{j\omega})$.

(d) Determine the sum

$$\sum_{k=-\infty}^{\infty} (-1)^n 0.5^{|n|}$$

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answer

Pr. 10.3 As shown in Example 1, when $\alpha = 0.5$ the DTFT of $x[n] = 0.5^{|n|}$ is

$$X(e^{j\omega}) = \frac{3/4}{5/4 - \cos(\omega)}$$

(a) If we let $\omega = 0$ then

$$X(1) = \frac{3/4}{5/4 - 1} = 3 = \sum_k x[n]$$

(b) The inverse DTFT is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

if we let $n = 0$ we get that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) d\omega = x[0]$$

and so the given integral is $2\pi x[0] = 2\pi$.

(c) From the DTFT, $X(e^{j\omega})$ is real and since the denominator, i.e., $5/4 - \cos(\omega)$, is positive for $[-\pi, \pi)$ the phase $< X(e^{j\omega}) = 0$.

(d) If we let $\omega = \pi$ in the DTFT we obtain

$$\sum_n x[n](-1)^n = X(e^{j\pi n}) = \frac{3/4}{9/4} = \frac{1}{3}$$

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Exercise 2

Duality of DTFT

The DTFT of a discrete-time signal $x[n]$ is given as

$$X(e^{j\omega}) = e^{j\pi/4}\delta(\omega - 0.5\pi) + e^{-j\pi/4}\delta(\omega + 0.5\pi) - 2\pi e^{-j\pi/8}\delta(\omega - 0.71) - 2\pi e^{j\pi/8}\delta(\omega + 0.71)$$

(a) Is the signal $x[n]$ periodic? If so, indicate its period.
(b) Determine the signal $x[n]$, and verify your answer above.

answer

Pr. 10.5 (a) This is the DFT of two sinusoids one of frequency 0.5π rad. and the other 0.71 rad., the ratio of these frequencies is not a rational number (because of π) so $x[n]$ is not periodic.

(b) The signal is

$$x[n] = \frac{1}{\pi}\cos(0.5\pi n + \pi/4) - 2\cos(0.71n - \pi/8)$$

To verify this, express

$$x[n] = \frac{1}{\pi}0.5(e^{j(0.5\pi n+\pi/4)} + e^{-j(0.5\pi n+\pi/4)}) - 2 \times 0.5(e^{j(0.71\pi n-\pi/8)} + e^{-j(0.71\pi n-\pi/4)})$$

an using that the DTFT of $e^{j\omega_0 n}$ is $2\pi\delta(\omega - \omega_0)$ we get

$$X(e^{j\omega}) = e^{j\pi/4}\delta(\omega - 0.5\pi) + e^{-j\pi/4}\delta(\omega + 0.5\pi) - 2\pi(e^{-j\pi/8}\delta(\omega - 0.71) + e^{j\pi/8}\delta(\omega + 0.71)$$

Exercise 3

Sinusoidal form of DTFT

A triangular pulse is given by

$$t[n] = \begin{cases} 3+n & -2 \le n \le -1 \\ 3-n & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) The pulse can be written as

$$t[n] = \sum_{k=-\infty}^{\infty} A_k \delta[n-k]$$

Find the $\{A_k\}$ coefficients.

(b) Find a sinusoidal expression for the DTFT of $t[n]$—that is,

$$T(e^{j\omega}) = B_0 + \sum_{k=1}^{\infty} B_k \cos(k\omega)$$

Express the coefficients B_0 and B_k in terms of the A_k coefficients.

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answer

<u>**Pr. 10.14**</u> (a) $A_k = 3 - |k|$, for $-2 \le k \le 2$, 0 otherwise.

(b) Writing

$$t[n] = \sum_{k=-2}^{2} (3 - |k|)\delta[n-k]$$
$$= 3\delta[n] + \sum_{k=1}^{2}(3-k)(\delta[n+k] + \delta[n-k])$$

with a Z-transform

$$T(z) = 3 + \sum_{k=1}^{2}(3-k)(z^k + z^{-k})$$

so that the DTFT is

$$T(e^{j\omega}) = 3 + \sum_{k=1}^{2}(3-k)[e^{j\omega k} + e^{-j\omega k}]$$
$$= \underbrace{3}_{B_0} + \sum_{k=1}^{2} \underbrace{2(3-k)}_{B_k} \cos(k\omega)$$

for $k > 2$, $B_k = 0$.

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Exercise 4

Computations from DTFT definition

For simple signals it is possible to obtain some information on their DTFTs without computing them. Let

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$$

(a) Find $X(e^{j0})$ and $X(e^{j\pi})$ without computing the DTFT $X(e^{j\omega})$.

(b) Find

$$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

(c) Find the phase of $X(e^{j\omega})$. Is it linear?

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answer

<u>**Pr. 10.16**</u> (a) Using

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\omega}$$

we have

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 1+2+3+2+1 = 9$$

$$X(e^{j\pi}) = \sum_{n=-\infty}^{\infty} (-1)^n x[n] = 1-2+3-2+1 = 1$$

(b) By Parseval's result

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = \sum_n |x[n]|^2$$

we get that

$$\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 2\pi(1+4+9+4+1) = 38\pi$$

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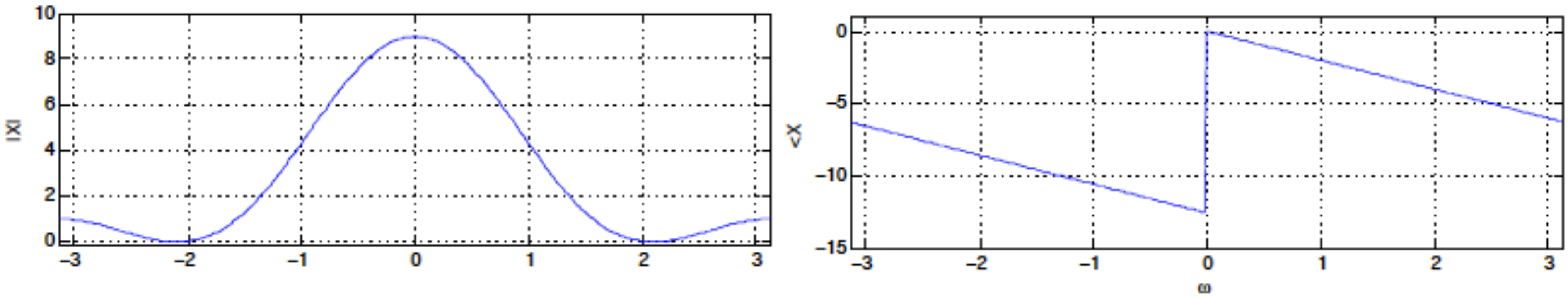
(c) The z-transform of $x[n]$ is

$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} = z^{-2}[z^2 + 2z^1 + 3 + 2z^{-1} + z^{-2}]$$

and the DTFT is

$$X(e^{j\omega}) = e^{-j2\omega}\left[\underbrace{3 + 2\cos(\omega) + 4\cos(2\omega)}_{\text{real-valued}}\right]$$

so that the phase is -2ω when above term is positive and $-2\omega \pm \pi$ when negative.



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Exercise 5

DTFT and Z-transform—MATLAB

Let $x[n] = r[n] - r[n-3] - u[n-3]$ where $r[n]$ is the ramp signal.

(a) Carefully plot $x[n]$ and find its Z-transform $X(z)$.

(b) If $y[n] = x[-n]$, give $Y(z)$ in terms of $X(z)$.

(c) Use the above results to find the DTFT of $x[n]$, $x[-n]$, and $x[n] + x[-n]$. Find the magnitude of each of these DTFTs and then use MATLAB to compute them and plot them.

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answer

Pr. 10.15 (a) The signal $x[n]$ can be written as

$$x[n] = \begin{cases} 0 & n < 0 \\ n & n = 0, 1, 2 \\ 0 & n \geq 3 \end{cases}$$

or $x[n] = \delta[n-1] + 2\delta[n-2]$; its Z-transform is

$$X(z) = z^{-1} + 2z^{-2}$$

(b) $y[n] = x[-n] = \delta[n+1] + 2\delta[n+2]$ so that

$$Y(z) = X(z^{-1}) = z + 2z^2$$

(c) The DTFTs of $x[n]$, $y[n] = x[-n]$ and $z[n] = x[n] + x[-n]$ are from the above

$$X(e^{j\omega}) = e^{-j\omega} + 2e^{-j2\omega}$$
$$Y(e^{j\omega}) = e^{j\omega} + 2e^{j2\omega}$$
$$Z(e^{j\omega}) = 2\cos(\omega) + 4\cos(2\omega)$$

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To compute their magnitude we consider

$$|X(e^{j\omega})|^2 = X(e^{j\omega})\underbrace{Y(e^{j\omega})}_{X(e^{-j\omega})} = 1 + 2e^{-j\omega} + 2e^{j\omega} + 4 = 5 + 4\cos(\omega)$$

so that $|X(e^{j\omega})| = \sqrt{5 + 4\cos(\omega)}$. We then have that

$$|Y(e^{j\omega})|^2 = Y(e^{j\omega})\underbrace{X(e^{j\omega})}_{Y(e^{-j\omega})}$$

so that $|Y(e^{j\omega})| = |X(e^{j\omega})|$. Finally,

$$|Z(e^{j\omega})| = |2\cos(\omega) + 4\cos(2\omega)|$$

because it is real.

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Exercise 6

DTFT of even and odd functions

A signal

$$x[n] = 0.5^n u[n]$$

is neither even nor odd.

(a) Find the even $x_e[n]$ and the odd $x_o[n]$ components of $x[n]$, and carefully plot them.

(b) Find the Z-transforms of $x_e[n]$ and $x_o[n]$, and from them find the DFTs $X_e(e^{j\omega})$ and $X_o(e^{j\omega})$. Are they real or imaginary?

(c) Since $x[n] = x_e[n] + x_o[n]$ so that $X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$, how dos the real and the imaginary parts of $X(e^{j\omega})$ relate to $X_e(e^{j\omega})$ and $X_o(e^{j\omega})$? Explain.

(d) Use Parseval's result to obtain that $E_x = E_{xe} + E_{xo}$ i.e., the energy of the signal is the sum of the energies of its even and odd components.

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answer

Pr. 10.17(a)–(c)The even and odd components of $x[n]$ are

$$x_e[n] = 0.5(x[n] + x[-n])$$
$$x_o[n] = 0.5(x[n] - x[-n])$$

and so their DTFTs are

$$X_e(e^{j\omega}) = 0.5[X(e^{j\omega}) + X(e^{-j\omega})]$$
$$X_o(e^{j\omega}) = 0.5[X(e^{j\omega}) - X(e^{-j\omega})]$$

The Z-transform of $x[n] = 0.5^n u[n]$ is $X(z) = 1/(1 - 0.5z^{-1})$ and its region of convergence include the unit circle. So the DTFT is

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1-0.5e^{-j\omega}} = \frac{1-0.5e^{j\omega}}{(1-0.5e^{-j\omega})(1-0.5e^{j\omega})} \\ &= \underbrace{\frac{(1-0.5\cos(\omega))}{1.25-\cos(\omega)}}_{\text{real part}} + j\underbrace{\frac{-0.5\sin(\omega)}{1.25-\cos(\omega)}}_{\text{imaginary part}} \end{aligned}$$

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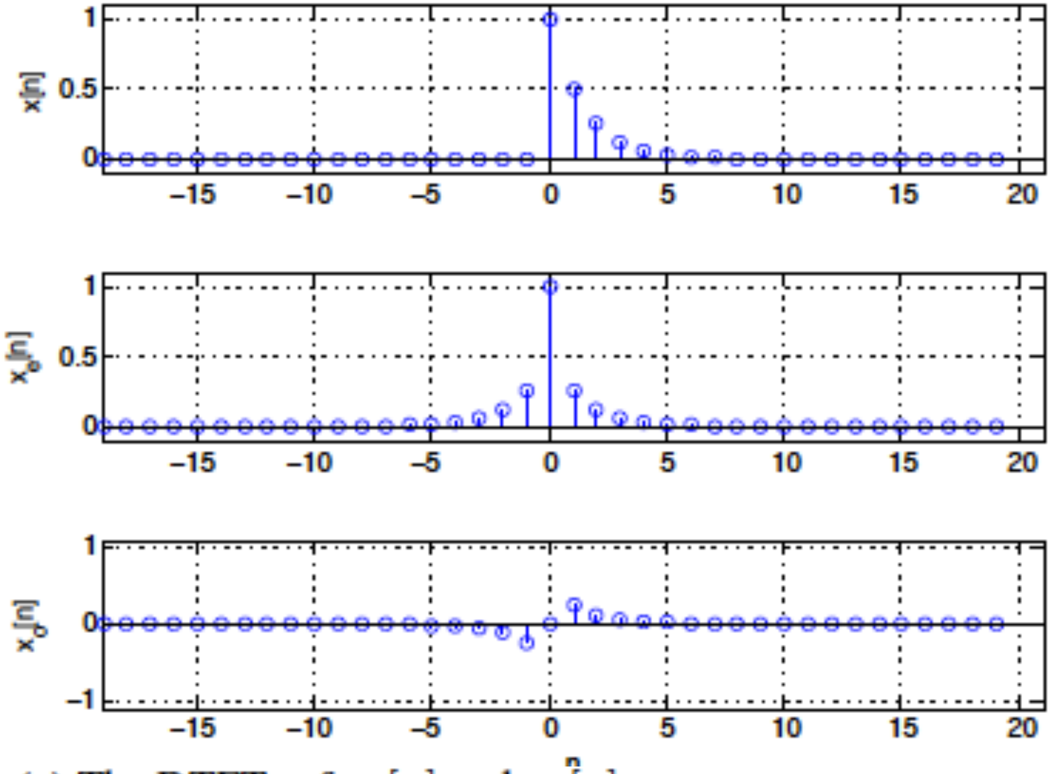

(c) The DTFTs of $x_e[n]$ and $x_o[n]$ are

$$X_e(e^{j\omega}) = \frac{0.5}{1-0.5e^{-j\omega}} + \frac{0.5}{1-0.5e^{j\omega}} = \frac{0.5(2-\cos(\omega))}{1.25-\cos(\omega)}$$

$$X_o(e^{j\omega}) = \frac{0.5}{1-0.5e^{-j\omega}} - \frac{0.5}{1-0.5e^{j\omega}} = \frac{-0.5j\sin(\omega)}{1.25-\cos(\omega)}$$

where the top one is real and the second imaginary. So that $X_e(e^{j\omega})$ is the real part of $X(e^{j\omega})$ and $X_o(e^{j\omega})$ is j times the imaginary part of $X(e^{j\omega})$.

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(d) Using Parseval's result, the energy of $x[n]$ is

$$\begin{aligned} E_x &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[|X_e(e^{j\omega})|^2 + |X_o(e^{j\omega})|^2 + X_e(e^{j\omega})X_o^*(e^{j\omega}) + X_e^*(e^{j\omega})X_o(e^{j\omega})\right] d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[|X_e(e^{j\omega})|^2 + |X_o(e^{j\omega})|^2\right] d\omega \\ &= E_{xe} + E_{xo} \end{aligned}$$

where the last two terms correspond to the energy of the even and odd components of $x[n]$. This is because the integral of the cross terms is zero.

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Exercise 7

Convolution sum and product of polynomials

The convolution sum can be seen as a way to compute the coefficients of the product of polynomials. This is because

$$[x * y][n] \Leftrightarrow X(z)Y(z) \Leftrightarrow X(e^{j\omega})Y(e^{j\omega})$$

(a) Let $X(z) = 1 + 2z^{-1} + 3z^{-2}$ and $Y(z) = z^{-2} + 4z^{-3}$ if $x[n] = 1\delta[n] + 2\delta[n-1] + 3\delta[n-2]$ and $y[n] = 1\delta[n-2] + 4\delta[n-3]$ are sequences formed by the coefficients of the polynomials. Compute the convolution sum $[x * y][n]$ and compare it to the coefficients of the polynomial $Z(z) = X(z)Y(z)$, or $Z(e^{j\omega}) = X(e^{j\omega})Y(e^{j\omega})$.

(b) Suppose that the transfer function of a discrete-time system is

$$H(z) = \frac{W(z)}{V(z)} = 3z^2 + 2z + 2z^{-1} + 3z^{-2}$$

and that it is known that the input is $v[n] = u[n] - u[n-3]$. Use the connection between the product of the polynomials and the convolution sum to find the output $w[n]$ of the system.

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answer

Pr. 10.19 (a) The convolution sum is given

$$z[n] = \sum_{k=0}^{n} x[k]y[n-k]$$

$$z[0] = x[0]y[0] = 1 \times 0 = 0$$

$$z[1] = x[0]y[1] + x[1]y[0] = 1 \times 0 + 2 \times 2 = 0$$

$$z[2] = x[0]y[2] + x[1]y[1] + x[2]y[0] = 1 \times 1 + 0 + 0 = 1$$

$$z[3] = x[0]y[3] + x[1]y[2] + x[2]y[1] + x[3]y[0] = 1 \times 4 + 2 \times 1 + 0 + 0 = 6$$

$$z[4] = x[0]y[4] + x[1]y[3] + x[2]y[2] + x[3]y[1] + x[4]y[0] = 1 \times 0 + 2 \times 4 + 3 \times 1 + 0 + 0 = 11$$

$$z[5] = x[0]y[5] + x[1]y[4] + x[2]y[3] + x[3]y[2] + x[4]y[1] + x[5]y[0] = 1 \times 0 + 2 \times 0 + 3 \times 4 + 0 + 0 + 0 = 12$$

$$z[n] = 0 \qquad n > 5$$

or $z[n] = 1\delta[n-2] + 6\delta[n-3] + 11\delta[n-4] + 12\delta[n-5]$. Multiplying the polynomials $X(z)$ by $Y(z)$ we get

$$Z(z) = 0 + 0z^{-1} + 1\,z^{-2} + (2+4)z^{-3} + (3+8)z^{-4} + 12z^{-5} = z^{-2} + 6z^{-3} + 11z^{-4} + 12z^{-5}$$

which is the Z-transform of the above $z[n]$ found by convolution.

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(b) The Z-transform of $v[n]$ is $V(z) = 1 + z^{-1} + z^{-2}$ so that

$$W(z) = (3z^2 + 2z + 2z^{-1} + 3z^{-2})V(z) = 3z^2 + 5z + 5 + 4z^{-1} + 5z^{-2} + 5z^{-3} + 3z^{-4}$$

so that

$$w[n] = 3\delta[n+2] + 5\delta[n+1] + 5\delta[n] + 4\delta[n-1] + 5\delta[n-2] + 5\delta[n-3] + 3\delta[n-4]$$