Ch.11 The Discrete-Time Fourier Transform (DTFT)

Contents

- definition of the DTFT
- relation to the z-transform, region of convergence, stability
- frequency plots
- convolution property, filters
- inverse DTFT

Skip sections 11.2.4.1 (decimation/interpolation), 11.3 (Fourier Series), 11.4 (DFT).

These are covered in EE2S31.

Discrete-time Fourier Transform

Definition of the Discrete-time Fourier transform (DTFT)

$$X(\omega) = \mathcal{F}\{x[n]\} := \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- Continuous function of ω (while $x[n]$ is a time series)
- $X(\omega + 2\pi) = X(\omega)$: periodic in ω, period 2π:

  It sufficies to consider the interval $\omega \in [-\pi, \pi]$.
- $X(\omega)$ is called "the spectrum" and measures the frequency content of $x[n]$.

  $X(\omega) = |X(\omega)| e^{j\phi(\omega)}$,

  where $|X(\omega)|$: amplitude spectrum, $\phi(\omega)$: phase spectrum
- Sufficient condition for convergence of the infinite sum:

$$\left|\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} |x[n]|\, |e^{-j\omega n}| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

  i.e., $x[n]$ is absolutely summable ($x \in \ell_1$).

Discrete-time Fourier Transform

Relation of the z-transform to the DTFT

The relation to the z-transform is obtained by setting $z = e^{j\omega}$ (assuming that the unit circle $|z| = 1$ is in de ROC).

Hence, we often write $X(\omega)$ as $X(e^{j\omega})$, cf. the book. (This notation avoids confusion between $X(\omega)$ and $X(z)$, different functions.)

We immediately obtain (LTI systems):

- $y[n] = h[n] * x[n] \quad \leftrightarrow \quad Y(\omega) = H(\omega)X(\omega)$ (filters!)
- $H(\omega) = \sum h[n]e^{-j\omega n}$ exists if the system is BIBO stable ($h \in \ell_1$, i.e., the unit circle is in the ROC of $H(z)$).
- $\delta[n] \quad \leftrightarrow \quad 1$

  $u[n] \quad \leftrightarrow$ (no ordinary DTFT because of ROC)

  $a^n u[n] \quad (|a| < 1) \quad \leftrightarrow \quad \dfrac{1}{1 - ae^{-j\omega}}$

Discrete-time Fourier Transform

Frequency plots

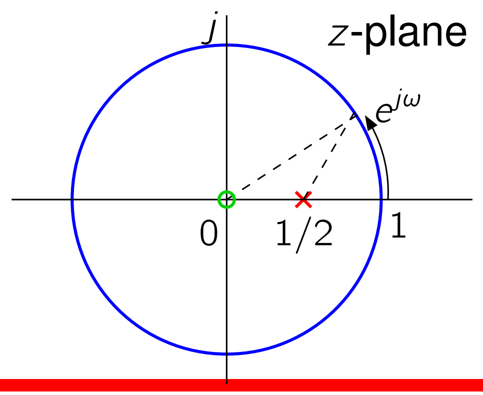
Plot the amplitude and phase spectrum of $X(\omega) = \dfrac{1}{1 - ae^{-j\omega}}$ (assume $a \in \mathbb{R}$)

- Amplitude spectrum

$$|X(\omega)|^2 = X(\omega)X^*(\omega) = \frac{1}{1 - ae^{-j\omega}} \frac{1}{1 - ae^{j\omega}} = \frac{1}{1 + a^2 - a(e^{j\omega} + e^{-j\omega})} = \frac{1}{1 + a^2 - 2a\cos(\omega)}$$

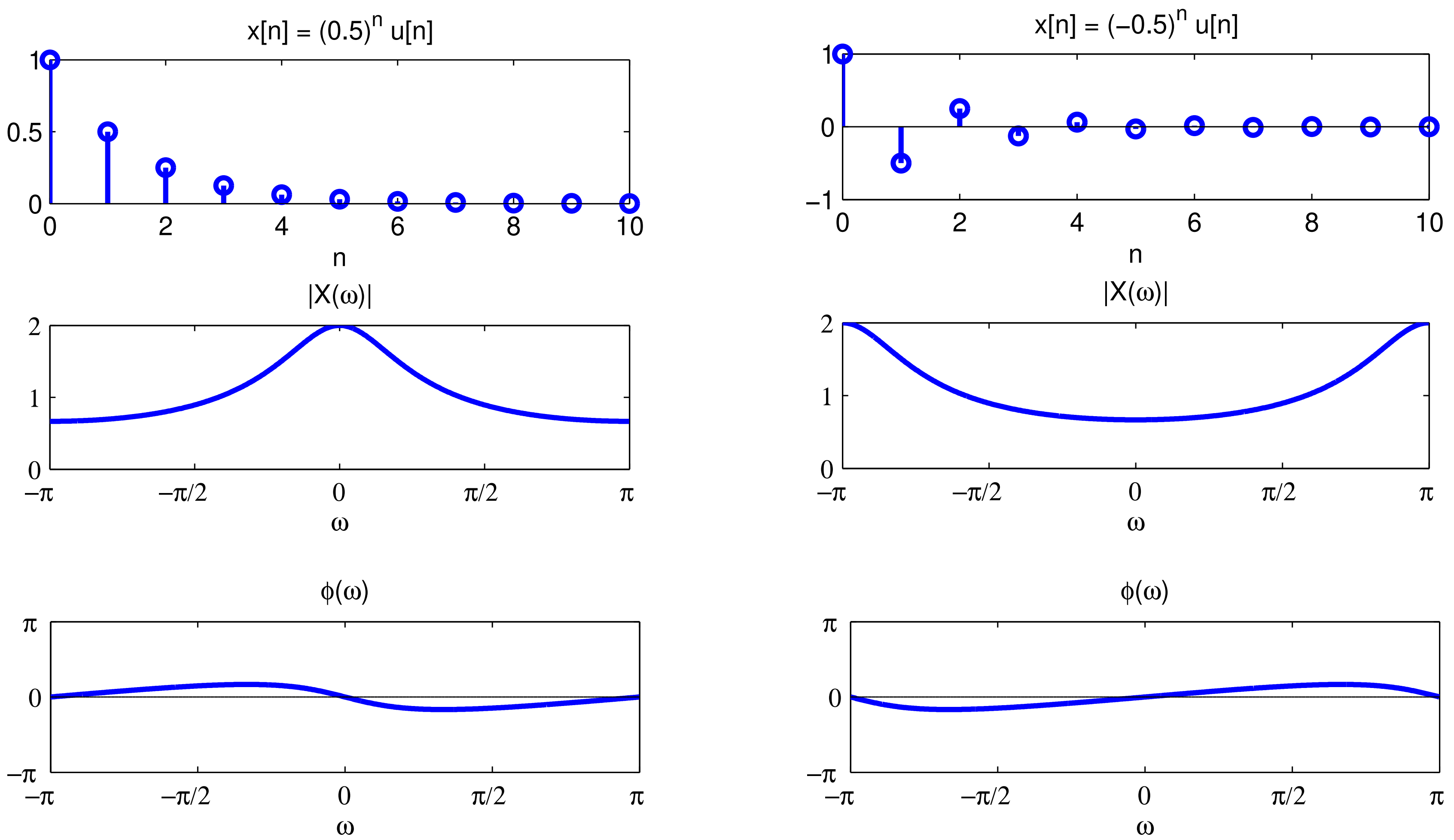
- Phase spectrum

$$\frac{1}{1 - ae^{-j\omega}} = \frac{1}{(1 - a\cos(\omega)) + ja\sin(\omega)} \quad \Rightarrow \quad \phi(\omega) = -\tan^{-1}\left(\frac{a\sin(\omega)}{1 - a\cos(\omega)}\right)$$



The response can also be estimated using phasors.

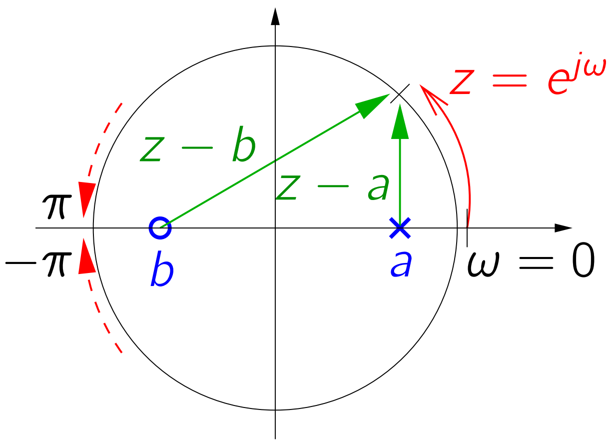
Discrete-time Fourier Transform



In matlab: use `atan2` or `angle`, use `unwrap` to resolve phase jumps of 2π.

Estimating frequency plots using phasors

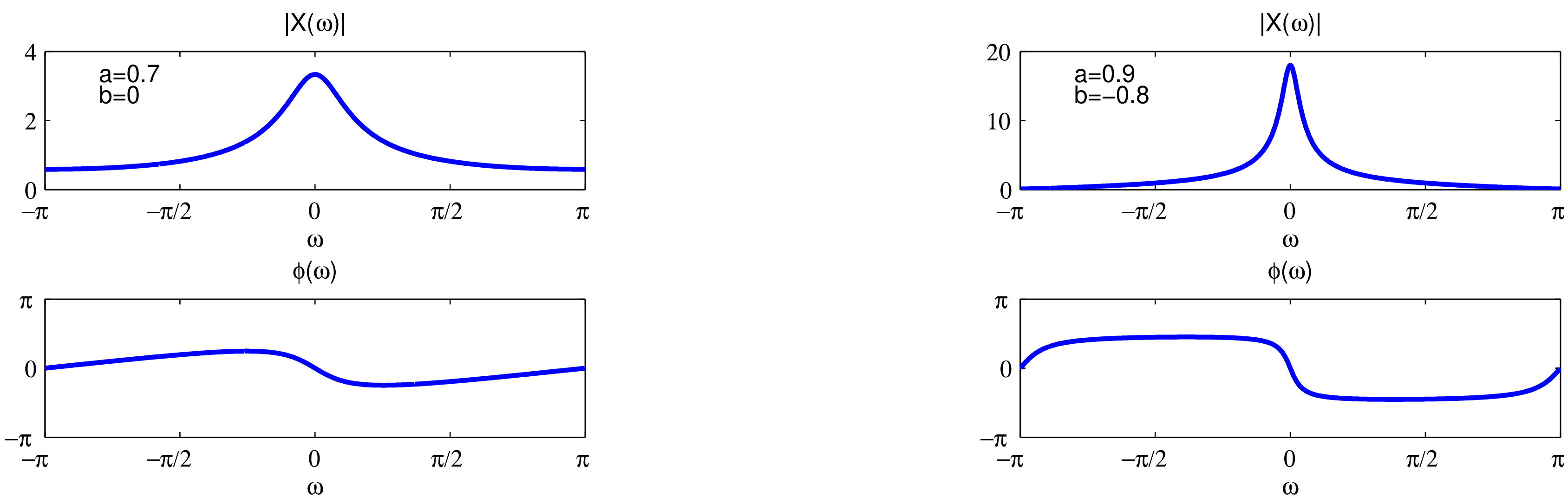
Given a transfer function $X(z) = \frac{z-b}{z-a}$, draw a plot of $|X(\omega)|$ and $\phi(\omega)$.



$$|X(\omega)| = \frac{|z-b|}{|z-a|}$$

$$\phi(\omega) = \angle(z-b) - \angle(z-a) \mod 2\pi$$

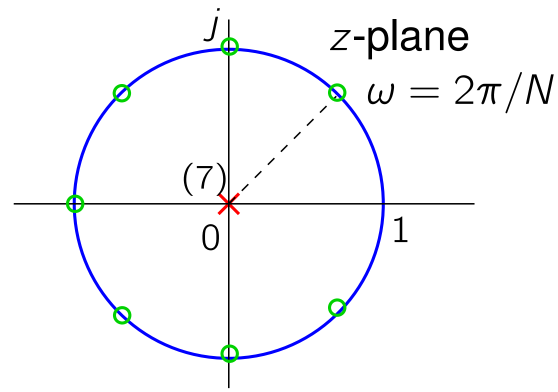
To gain some insight: compute this for a number of values of ω.



Discrete-time Fourier Transform

Example: DTFT of a pulse

$$p[n] = u[n] - u[n-N]\,, \qquad \text{pulse of length } N$$



$$P(z) = 1 + z^{-1} + \cdots + z^{-(N-1)} = \frac{1 - z^{-N}}{1 - z^{-1}}$$

$$P(\omega) = \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} = \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} e^{-j\omega(N-1)/2} = \frac{\sin(\omega N/2)}{\sin(\omega/2)} e^{-j\omega(N-1)/2}$$

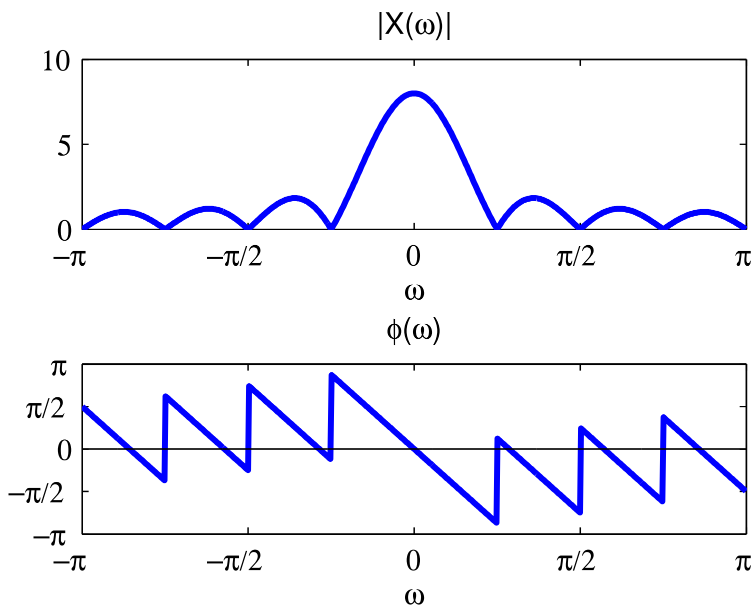
- The amplitude spectrum is

$$A(\omega) = |P(\omega)| = \left| \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right| \,, \qquad \text{"periodic sinc-function" (Dirichlet-function)}$$

with $A(0) = N$ (L'Hopital).

- The phase spectrum is $\phi(\omega) = -\omega(N-1)/2$ (lineair phase) plus phase jumps of π due to sign changes of $\sin(\omega N/2)$.

Discrete-time Fourier Transform



$(N = 8)$

zero crossings for $\omega = \pm\frac{2\pi}{N}k$ $(k \neq 0)$

Phase slope $\leftrightarrow$ delay

Phase jumps (π) $\leftrightarrow$ change of sign

The linear phase corresponds to a delay $z^{-(N-1)/2}$, half the duration of the pulse.

The first zero in the amplitude spectrum (right of the peak at $\omega = 0$) gives an indication of the "bandwidth" (although this signal is not band limited):

$$\Delta\omega = \frac{2\pi}{N}$$

Discrete-time Fourier Transform

Note:

- The duration of the pulse (N) is inversely proportional to the “bandwidth” in the frequency domain ($2\pi/N$).
- If $N = 1$ then $p[n] = \delta[n]$, and $P(\omega) = 1$ (constant).
- If $N \rightarrow \infty$ then $p[n] \rightarrow u[n]$ (step) and $\Delta\omega \rightarrow 0$: the amplitude spectrum converges to an impulse.

  (Not allowed to take the limit because the phase does not converge – see later for the DTFT of $u[n]$.)

Discrete-time Fourier Transform

Example: DTFT of a non-causal signal

Determine the spectrum of the non-causal signal $x[n] = a^{|n|}$ with $|a| < 1$.

Solution

The z-transform of $x[n]$ is (transform the causal/anti-causal parts separately):

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^n z^n - 1 = \frac{1}{1 - az^{-1}} + \frac{1}{1 - az} - 1 = \frac{1 - a^2}{1 - a(z + z^{-1}) + a^2}$$

with as ROC the intersection of the ROC of the causal and anti-causal part:

$$\text{ROC: } |a| < |z| < \frac{1}{|a|}$$

The ROC contains the unit circle. Hence

$$X(\omega) = X(z = e^{j\omega}) = \frac{1 - a^2}{1 - a(e^{j\omega} + e^{-j\omega}) + a^2} = \frac{1 - a^2}{1 + a^2 - 2a\cos(\omega)}$$

Note that $X(\omega)$ is real-valued ($\phi(\omega) = 0$). We have seen the plot of $|X(\omega)|$ before...

Discrete-time Fourier Transform

Relation of the continuous-time Fourier Transform to the DTFT

Consider a signal $x(t)$ and sample it with period T_s,

$$x_s(t) = \sum_n x(nT_s)\delta(t - nT_s)$$

The (continous-time) Fourier transform is

$$X_s(\Omega) = \mathcal{F}\{x_s(t)\} = \sum_n x(nT_s)\mathcal{F}\{\delta(t - nT_s)\} = \sum x(nT_s)e^{-jn\Omega T_s}$$

Set $\omega = \Omega T_s$ and $x[n] = x(nT_s)$. Then

$$X_s(\Omega) = \mathcal{F}\{x_s(t)\} = \sum_n x[n]e^{-jn\omega} =: X(\omega)$$

The definition of $X(\omega)$ (spectrum of a time series) is consistent to that of $X_s(\Omega)$ (spectrum of a continuous-time signal).

Discrete-time Fourier Transform

Inverse DTFT

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}\mathrm{d}\omega = \int_{-1/2}^{1/2} X(f)e^{j2\pi f n}\mathrm{d}f \quad (\text{with } \omega = 2\pi f)$$

The integral runs over 1 period of the spectrum.

Proof:

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k}\right]e^{j\omega n}\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\left[\int_{-\pi}^{\pi} e^{j\omega(n-k)}\mathrm{d}\omega\right] \\
&= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} x[k]\cdot 2\pi\delta[n-k] \quad = \quad x[n]
\end{aligned}$$

- The same result is obtained by considering $X(\omega)$ $(-\pi \le \omega \le \pi)$ as the spectrum of the continuous-time signal $x(t)$, computing the corresponding $x(t)$ (Inverse FT), and sampling with $t = nT_s$.

Discrete-time Fourier Transform

Energy (Parseval)

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

$S_x(\omega) := |X(\omega)|^2$ is called the energy spectrum ("energy spectral density": energy per radial)

Proof

$$\begin{aligned}
E_x &= \sum_n |x[n]|^2 = \sum_n x[n]x^*[n] \\
&= \sum_n x[n]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n}d\omega\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_n x[n]e^{-j\omega n}\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega
\end{aligned}$$

- A similar property for power (see book) seems less practical...

Discrete-time Fourier Transform

A *sufficient* condition for the existence of the DTFT was that the signal is absolutely summable. But also for some other signals we can define the DTFT.

Extension to signals with finite energy

Signals with finite energy ($x \in \ell_2$) are not always absolutely summable (the reverse does hold: $\ell_1 \subset \ell_2$). Due to Parseval, the spectrum has equal energy: also finite. We can define a DTFT pair (signal/spectrum) based on the Inverse DTFT (integral over a finite interval).

Example:

Ideal low-pass filter:

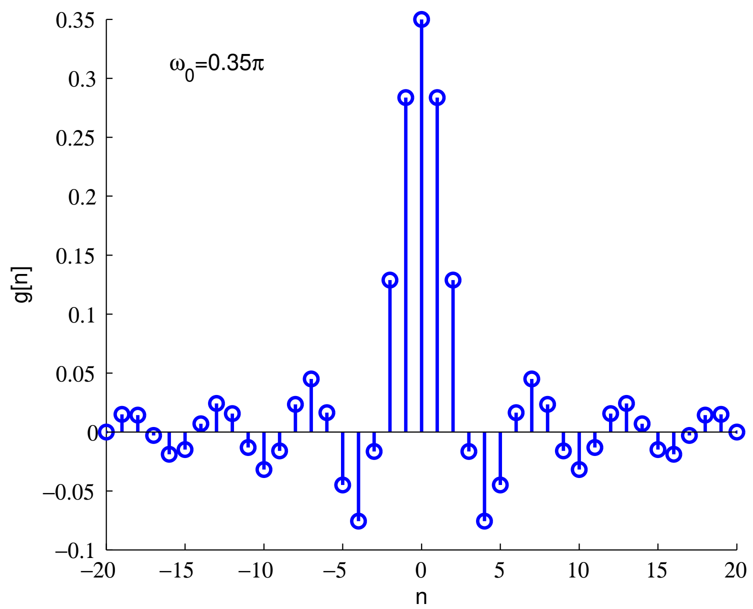
$$G(\omega) = \begin{cases} 1, & -\omega_0 \le \omega \le \omega_0\,, \qquad \text{with copies every } 2\pi k \\ 0, & \text{elsewhere} \end{cases}$$

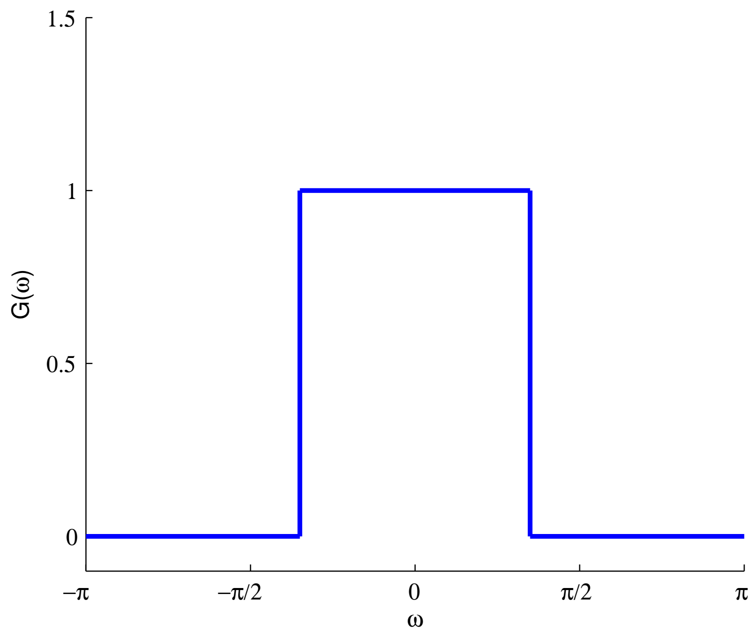
Discrete-time Fourier Transform

$$g[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(\omega)e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{\sin(\omega_0 n)}{\pi n}$$

$g[n]$ has finite energy but is not absolutely summable (because $\frac{1}{n}$ converges to 0 very slowly)





Discrete-time Fourier Transform

Further extension to non-absolutely summable signals

According to the equation, the Inverse DTFT of $2\pi\delta(\omega-\omega_0)$ equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega-\omega_0)\, e^{j\omega n}\mathrm{d}\omega \;=\; e^{j\omega_0 n}$$

Hence

$$e^{j\omega_0 n} \qquad \leftrightarrow \qquad 2\pi\delta(\omega-\omega_0)$$

This can be used to compute the DTFT of some signals which are not absolutely summable nor have finite energy (with impulses in the frequency domain), e.g., periodic signals.

- $x[n] = A \quad (-\infty < n < \infty)$ (constant signal) is not absolutely summable. The DTFT is

  $$X(\omega) = 2\pi A\delta(\omega)\,, \qquad -\pi \leq \omega < \pi$$

  Outside this interval: periodic (period 2π), or $X(\omega) = 2\pi A\sum_k \delta(\omega - 2\pi k)$.

Discrete-time Fourier Transform

- The DTFT of $x[n] = \cos(\omega_0 n + \theta) = \frac{1}{2}\left[e^{j(\omega_0 n+\theta)} + e^{-j(\omega_0 n+\theta)}\right]$ is

$$X(\omega) = \pi\left[e^{j\theta}\delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\right], \qquad -\pi \le \omega < \pi$$

Outside this interval: periodic (period 2π).

- More in general, consider

$$x[n] = \sum_\ell A_\ell \cos(\omega_\ell n + \theta_\ell) \quad \leftrightarrow \quad X(\omega) = \sum_\ell \pi A_\ell \left[e^{j\theta_\ell}\delta(\omega-\omega_\ell) + e^{-j\theta_\ell}\delta(\omega+\omega_\ell)\right]$$

for $-\pi \le \omega < \pi$ (periodic outside this interval).

A periodic signal $x[n]$ has harmonically related frequencies: $\omega_\ell = \ell\omega_0$, with $\omega_0 = \frac{2\pi}{N}$, where N is the period (in samples). We obtain a line spectrum, just like with the Fourier Series.

Discrete-time Fourier Transform

DTFT of a step

The z-transform of a unit step $u[n]$ is

$$\frac{1}{1 - z^{-1}}, \qquad \text{ROC: } |z| > 1$$

The unit cicle is not in the ROC, thus the DTFT can only be defined in generalized sense. ($u[n]$ is not absolutely summable and does not have finite energy.)

Define the discrete-time “sign” function:

$$\text{sgn}[n] = \begin{cases} 1\,, & n \geq 0 \\ -1\,, & n < 0 \end{cases}$$

Then

$$\text{sgn}[n] \leftrightarrow \frac{2}{1 - e^{-j\omega}}$$

$$u[n] = \frac{1}{2} + \frac{1}{2}\text{sgn}[n] \leftrightarrow \pi \sum_k \delta(\omega - 2\pi k) + \frac{1}{1 - e^{-j\omega}}$$

Discrete-time Fourier Transform

Proof (indication)

Using the Inverse DTFT:

$$\mathcal{F}^{-1}\left\{\frac{2}{1-e^{-j\omega}}\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{2e^{j\omega n}}{1-e^{-j\omega}}\mathrm{d}\omega = \text{(trig. tricks...)} = \text{sgn}[n]$$

Alternative proof

Using the Fourier transform of $\delta[n] = u[n] - u[n-1] = \frac{1}{2}(\text{sgn}[n] - \text{sgn}[n-1])$:

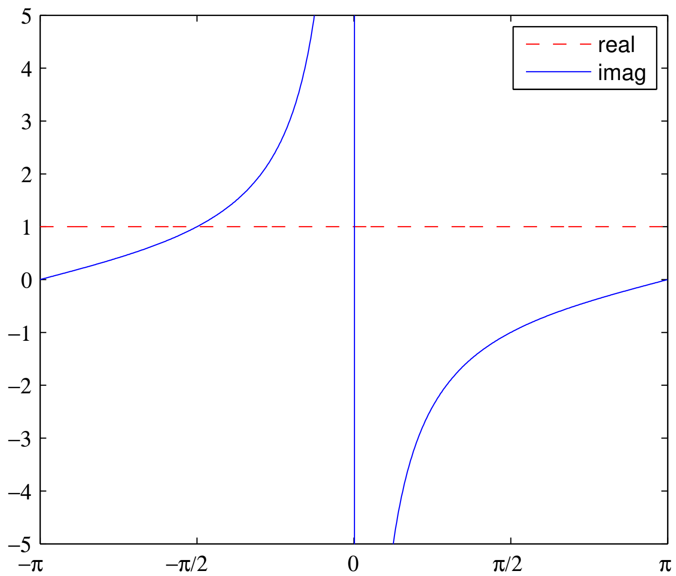
$$1 = \tfrac{1}{2}\mathcal{F}\{\text{sgn}[n]\} - \tfrac{1}{2}\mathcal{F}\{\text{sgn}[n-1]\} = \tfrac{1}{2}\mathcal{F}\{\text{sgn}[n]\} - \tfrac{1}{2}e^{-j\omega}\mathcal{F}\{\text{sgn}[n]\}$$

$$\Rightarrow \quad \mathcal{F}\{\text{sgn}[n]\} = \tfrac{2}{1-e^{-j\omega}} \qquad \text{for } \omega \neq \cdots, 0, 2\pi, 4\pi, \cdots$$

For $\omega = \cdots, 0, 2\pi, 4\pi, \cdots$ we consider the DC component of the function, which equals 0 (in contrast to $u[n]$, which motivates why we looked at sng$[n]$).

$\mathcal{F}\{u[n]\}$ has impulses at these frequencies.

Plot



$$\frac{2}{1 - e^{-j\omega}} = 1 - j\frac{\sin(\omega)}{1 - \cos(\omega)}$$

The real-valued part ‘1’ is constant, because we defined $\text{sgn}[0] = 1$ instead of 0.

Compare this to the Fourier transform of a continuous-time step function:

$\mathcal{F}\{u(t)\} = \frac{1}{j\Omega} + \pi\delta(\Omega)$.

Discrete-time Fourier Transform

Shift in time

If

$$x[n] \qquad \leftrightarrow \qquad X(\omega) = |X(\omega)| \cdot e^{j\theta(\omega)}$$

and $y[n] = x[n-N]$ is a delay by N samples, then

$$Y(\omega) = \sum_n x[n-N]e^{-j\omega n} = e^{-j\omega N}X(\omega)$$

so that

$$Y(\omega) = |X(\omega)| \cdot e^{j(\theta(\omega)-\omega N)}$$

- The delay only affects the phase (which drops with a negative slope as function of ω). A linear phase term ($-\omega N$) corresponds to a delay.

Discrete-time Fourier Transform

Shift in frequency

If

$$Y(\omega) = X(\omega - \omega_0)$$

is a frequency shift of $X(\omega)$ by ω_0, then

$$y[n] = x[n] \cdot e^{j\omega_0 n}$$

$y[n]$ equals $x[n]$ modulated by a complex exponential function $e^{j\omega_0 n}$.

- Likewise:

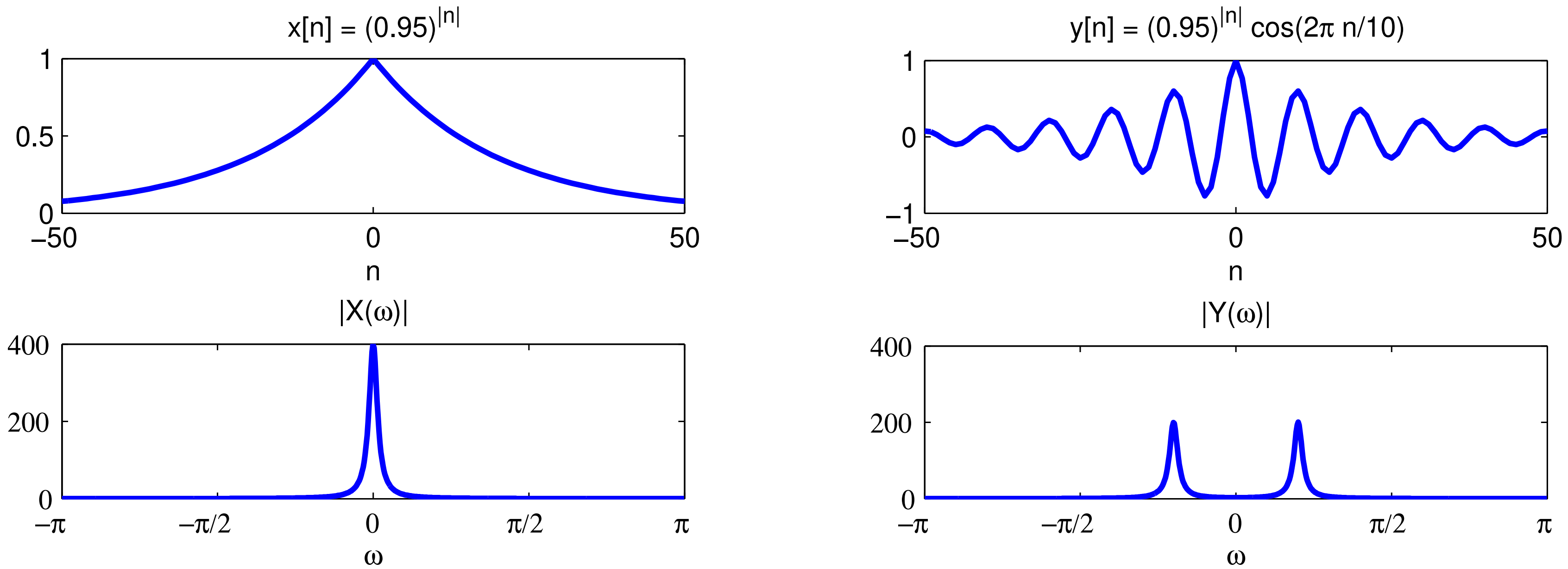
$$x[n] \cdot \cos(\omega_0 n) \quad \leftrightarrow \quad \frac{1}{2}\left[X(\omega - \omega_0) + X(\omega + \omega_0)\right]$$

$$x[n] \cdot \sin(\omega_0 n) \quad \leftrightarrow \quad \frac{-j}{2}\left[X(\omega - \omega_0) - X(\omega + \omega_0)\right]$$

The modulation shifts the spectrum of $x[n]$ to frequency ω_0.

Example of modulation

$$x[n] = a^{|n|}\cos(\omega_0 n) \qquad \text{with } a = 0.95,\ \omega_0 = \frac{2\pi}{10}$$



Discrete-time Fourier Transform

More generally: product of two signals

The DTFT of the product $x[n]y[n]$ is

$$\begin{aligned}\sum x[n]y[n]e^{-j\omega n} &= \sum \left[\frac{1}{2\pi}\int X(\theta)e^{j\theta n}\mathrm{d}\theta\right] y[n]e^{-j\omega n} \\ &= \frac{1}{2\pi}\int X(\theta)\left[\sum y[n]e^{-j(\omega-\theta)n}\right]\mathrm{d}\theta \\ &= \frac{1}{2\pi}\int X(\theta)Y(\omega-\theta)\mathrm{d}\theta\end{aligned}$$

Hence: a product in time becomes a convolution in frequency domain (dual to the previous result)

$$x[n]y[n] \quad \leftrightarrow \quad (X*Y)(\omega) := \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)Y(\omega-\theta)\mathrm{d}\theta$$

- Special case (seen before): $y[n] = x[n]e^{j\omega_0 n} \quad \leftrightarrow \quad Y(\omega) = X(\omega-\omega_0)$
  because $e^{j\omega_0 n} \leftrightarrow 2\pi\delta(\omega-\omega_0)$.

Discrete-time Fourier Transform

Real-valued signals

For real-valued signals, $x[n] = x^*[n]$. Hence

$$X^*(\omega) = X(-\omega) \quad = \quad X(z^{-1})\Big|_{z=e^{j\omega}}$$

and thus

$$|X(-\omega)| = |X(\omega)| \; : \quad \text{even in } \omega; \qquad \phi(-\omega) = -\phi(\omega) \; : \quad \text{odd in } \omega$$

It suffices to consider the spectrum on the interval $0 \le \omega \le \pi$.

Even real-valued signals

If moreover $x[n] = x[-n]$, then $X(\omega)$ is real-valued:

$$X^*(\omega) = \sum_{n=-\infty}^{\infty} x^*[n] e^{j\omega n} = \sum_{n=-\infty}^{\infty} x[-n] e^{j\omega n} = X(\omega)$$

The phase spectrum $\phi(\omega)$ is 0 except for jumps of π due to sign changes in $X(\omega)$.

Discrete-time Fourier Transform

Summary of properties (cf Table 11.1 p.750)

$$ax[n] + by[n] \quad \leftrightarrow \quad aX(\omega) + bY(\omega)$$

$$x[n-k] \quad \leftrightarrow \quad e^{-j\omega k}X(\omega)$$

$$x[-n] \quad \leftrightarrow \quad X(-\omega)$$

$$x^*[n] \quad \leftrightarrow \quad X^*(-\omega)$$

$$(x_1 * x_2)[n] \quad \leftrightarrow \quad X_1(\omega)X_2(\omega)$$

$$x[n]y[n] \quad \leftrightarrow \quad (X * Y)(\omega) = \tfrac{1}{2\pi}\int_{-\pi}^{\pi} X(\theta)Y(\omega-\theta)\mathrm{d}\theta$$

$$e^{j\omega_0 n} \quad \leftrightarrow \quad 2\pi\delta(\omega-\omega_0)$$

$$e^{j\omega_0 n}x[n] \quad \leftrightarrow \quad X(\omega-\omega_0)$$

$$x[n]\cos(\omega_0 n) \quad \leftrightarrow \quad \tfrac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right]$$

Parseval: $$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 \mathrm{d}\omega$$

(Note error in book regarding property (6), (7): $\cos(\omega_0 n)u[n]$ should be $\cos(\omega_0 n)$, and $\sin(\omega_0 n) \leftrightarrow -j\pi[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$.)

Discrete-time Fourier Transform

Discrete Fourier Transform (DFT)

Suppose $x[n]$ has a finite length of N samples (support $0 \leq n \leq N-1$), or $x[n]$ is periodic with period N, and we consider only 1 period.

The DTFT $X(\omega)$ is a continuous function of ω, with $-\pi \leq \omega < \pi$.

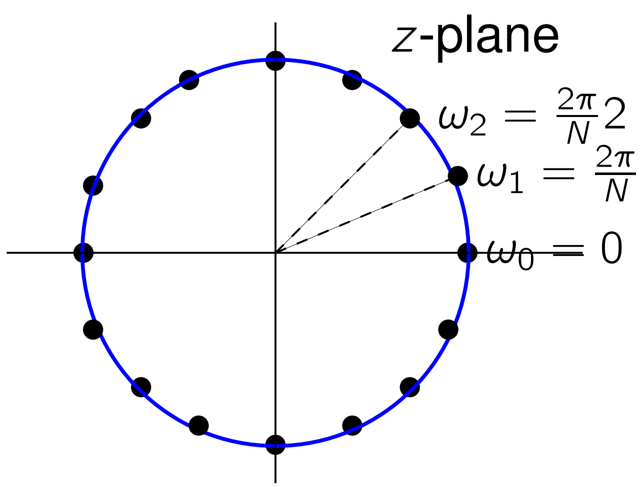
We sample $X(\omega)$ with N samples:

$$X[k] := X(\omega_k) \qquad \text{with} \qquad \omega_k = \frac{2\pi}{N}k, \quad k = 0, \cdots, N-1\,.$$

We obtain

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}kn}$$

$X[k]$ is called the Discrete Fourier Transform (DFT).



Discrete-time Fourier Transform

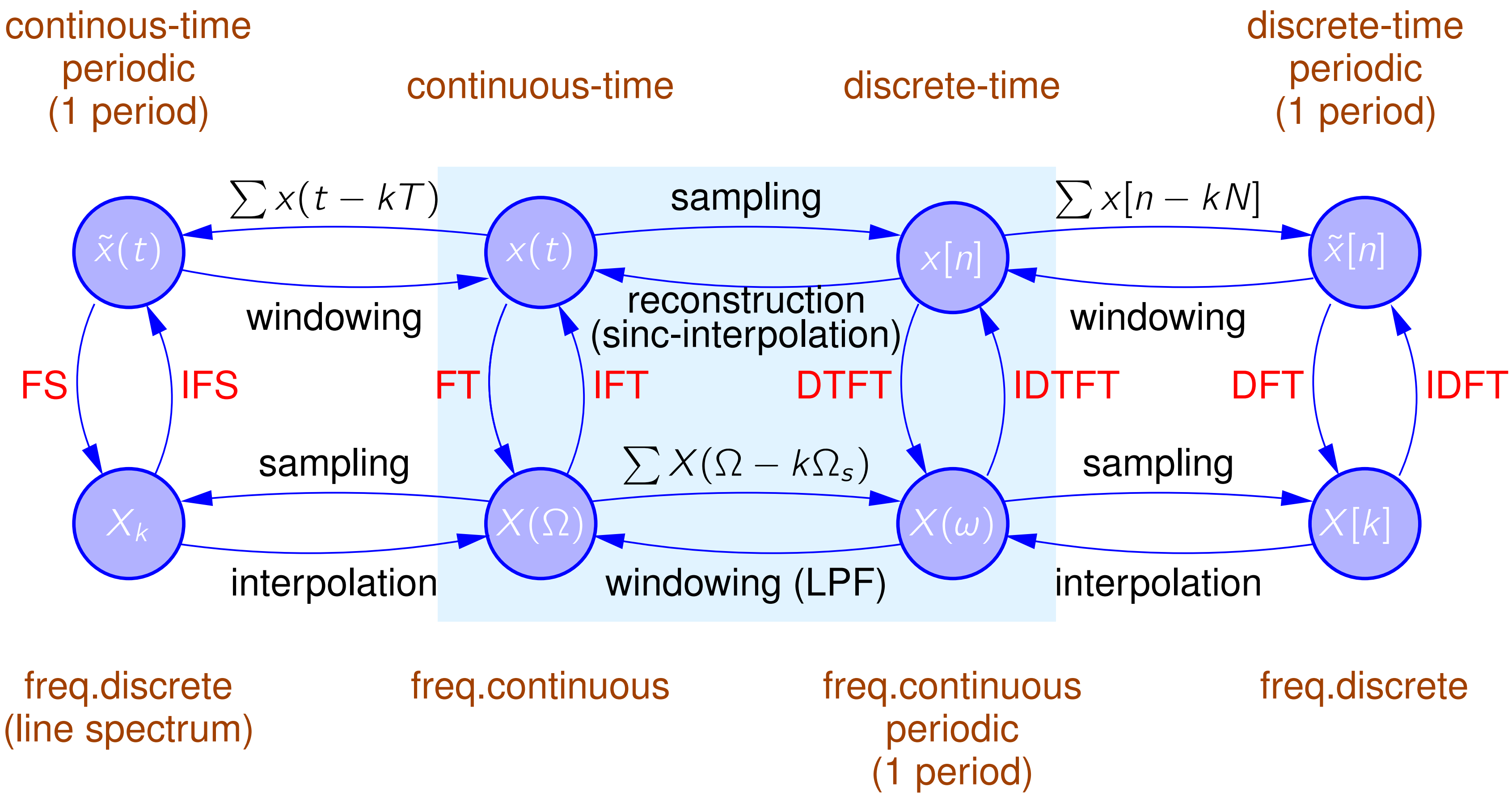
- N samples in frequency suffice to recover $x[n]$, $0 \le n \le N-1$ (outside this interval: periodic or zero)
- Compututionally efficient due to the Fast Fourier Transform (FFT)

For a periodic $x[n]$ with period N, this corresponds to a Fourier Series.

The DFT en its properties are discussed in EE2S31 (Q4), and in the practical of EE2T11 (Q3).

Discrete-time Fourier Transform

Relations



Generally:

- periodic $\leftrightarrow$ discrete
- short $\leftrightarrow$ long
- product $\leftrightarrow$ convolution