# Ch.10 The z-transform

**Contents**

- definition of the $z$-transform
- region of convergence
- convolution property, stability
- inverse $z$-transform

Skip sections 10.5.3, 10.6, 10.7

# The $z$-transform

## Sampled sequence

Suppose that we have a sampled signal:

$$x_s(t) = \sum x[n]\delta(t - nT_s)\,, \qquad x[n] := x(nT_s)$$

The Laplace transform $\mathcal{L}\{x_s(t)\}$ is

$$X_s(s) = \sum x[n]\mathcal{L}\{\delta(t - nT_s)\} = \sum x[n]e^{-sT_s n} \quad = \quad \sum x[n]z^{-n}\,,$$

where $z := e^{sT_s}$.

- For $s = j\Omega$ we obtain $z = e^{j\Omega T_s} = e^{j\omega}$, with $\omega = \Omega T_s$.
- More generally: $s = \sigma + j\Omega$ becomes $z = e^{\sigma T_s} e^{j\Omega T_s} = re^{j\omega}$.

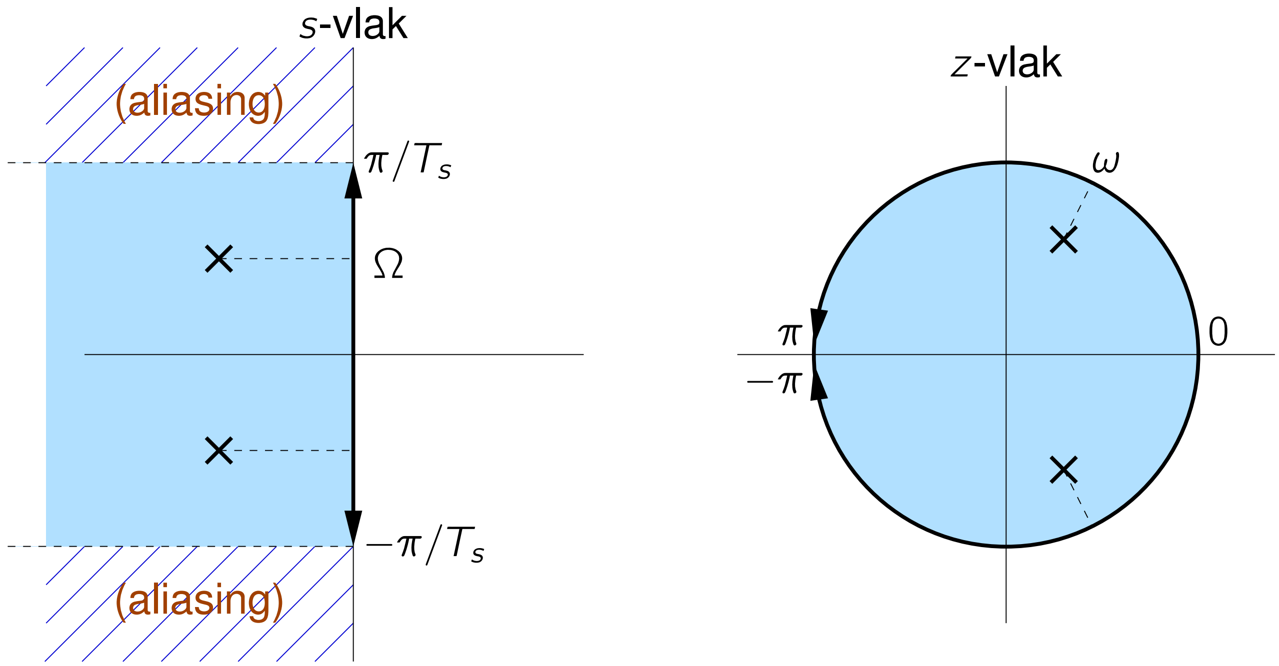
# The $z$-transform

Note that the mapping $s \rightarrow z = e^{sT_s}$ is not one-to-one.

For a given $z = e^{j\omega}$ we can take $-\pi \leq \omega \leq \pi$, this corresponds to
$-\frac{\pi}{T_s} \leq \Omega \leq \frac{\pi}{T_s}$: the fundamental interval.

Complex numbers $s = j\Omega$ with $\Omega$ outside this interval are mapped onto the same $z$.

Left half-plane is mapped to the inside of the unit circle.



# The $z$-transform

From now on, we will work with $z$ and apply this transform to time series, even if there is no connection to continuous-time signals.

## The $z$-transform

- The (two-sided) $z$-transform of a time series $x[n]$ is defined as

$$X(z) = \mathcal{Z}(x[n]) := \sum_{n=-\infty}^{\infty} x[n] z^{-n}, \qquad z \in \text{ROC}$$

We also need to indicate the region of convergence (ROC).

For example:

$$x = [\cdots, 0, 1, 2, \boxed{3}, 4, 5, 0, \cdots] \quad \Rightarrow \quad X(z) = z^2 + 2z^1 + 3 + 4z^{-1} + 5z^{-2}$$

ROC: $z \in \mathbb{C} \backslash \{0, \infty\}$

$$x[n] = a^n u[n] \quad \Rightarrow \quad X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

ROC: $|z| > a$

## The $z$-transform

- A few properties:

| | | | |
|---|---|---|---|
| $ax[n] + by[n]$ | $\Leftrightarrow$ | $aX(z) + bY(z)$ | |
| $x[n-k]$ | $\Leftrightarrow$ | $z^{-k}X(z)$ | |
| $a^n x[n]$ | $\Leftrightarrow$ | $X(\frac{z}{a})$ | often $a = e^{j\omega_0}$ (modulation) |
| $x[-n]$ | $\Leftrightarrow$ | $X(z^{-1})$ | |
| $x[n] = \delta[n]$ | $\Leftrightarrow$ | $X(z) = 1$ | ROC: $z \in \mathbb{C}$ |
| $x[n] = u[n]$ | $\Leftrightarrow$ | $X(z) = \frac{1}{1-z^{-1}}$ | ROC: $\lvert z\rvert > 1$ |

See Chaparro p.589-590 (2nd ed. p.653) for tables and more properties.

## Region of convergence

The region of convergence (ROC) of the $z$-transform of a signal $x[n]$ contains those values of $z$ for which the summation converges.

With $z = re^{j\omega}$ we find

$$\text{ROC:} \quad |X(z)| = |\sum x[n]z^{-n}| \leq \sum |x[n]|\, r^{-n} < \infty$$

- The ROC is the area where $|X(z)| < \infty$, this depends on $r$ but not on $\omega$. Hence, the ROC is limited by circles.
- $X(z)$ and the ROC together uniquely determine $x[n]$.
- Poles $p_k$ are the locations where $X(p_k) \to \infty$: these are never in the ROC.
  Zeros $z_k$ are the locations where $X(z_k) = 0$.

# The $z$-transform

## Example

- Determine the poles and zeros of

$$X(z) = 1 + 2z^{-1} \quad = \quad \frac{z+2}{z}$$

Answer: 1 pole at $z = 0$; 1 zero at $z = -2$.

- Same for

$$X(z) = \frac{1+2z^{-1}}{1+z^{-2}} \quad = \quad \frac{z(z+2)}{z^2+1}$$

Answer: poles at $z = \pm j$; 1 zero at $z = -2$, 1 zero at $z = 0$.

Theory says that for rational functions, the number of poles equals the number of zeros (also taking into account those at $z = 0$ and $z = \infty$).

If $X(z)$ is a rational function with real-valued coefficients, then the complex poles and zeros appear in conjugated pairs: if $p_k$ is a complex pole, then so is $p_k^*$.

## ROC for a finite sequence

If $x[n] = 0$ outside an interval $-\infty < N_0 \leq n \leq N_1 < \infty$, i.e.

$$X(z) = x[N_0]z^{-N_0} + \cdots + x[N_1]z^{-N_1}$$

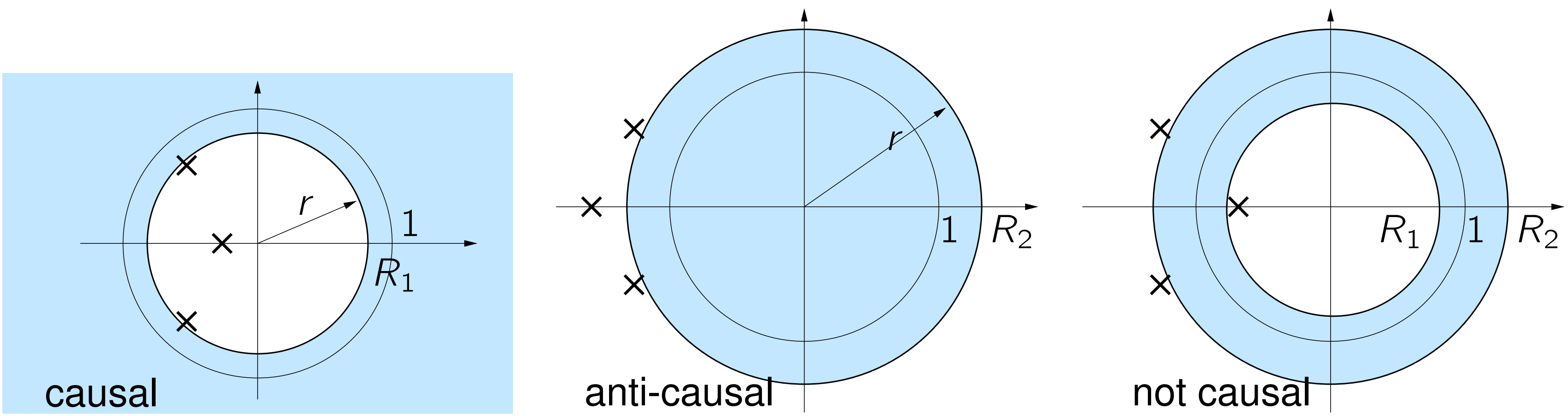
then the sum has a finite number of terms, and the ROC is all of $\mathbb{C}$, except perpaps at $z = 0$ or $|z| = \infty$:

$$\begin{array}{lll} 0\,, & \text{if } N_1 \geq 0\,, & \text{e.g.: } X(z) = z + 1 + z^{-1} \\ \infty\,, & \text{if } N_0 \leq 0 & \end{array}$$

## ROC of an infinite sequence

Split the sequence $x[n]$ into the sum of a causal and an anti-causal term, and use the linearity of the $z$-transform.

- The causal part $X_c(z)$ has ROC: $|z| > R_1$, the largest radius of the poles.
- The anti-causal part $X_{ac}(z)$ has ROC: $|z| < R_2$, the smallest radius of the poles.
- Hence, the ROC of $X(z)$ is the intersection: $R_1 < |z| < R_2$. All poles are outside the ROC.

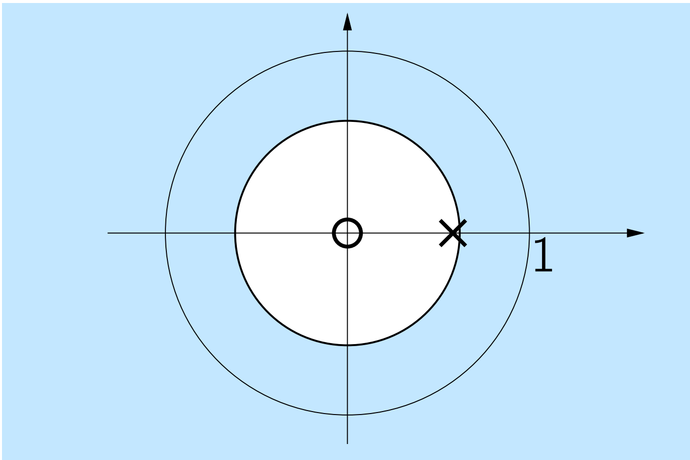


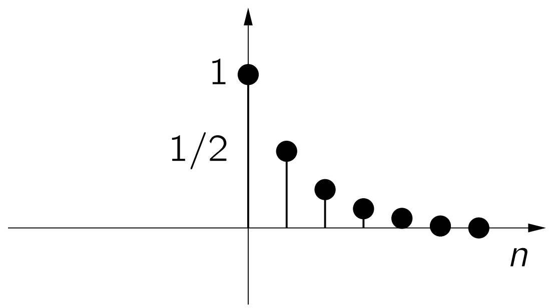
# The $z$-transform

## Example

- **Causal signal**

$$x_1[n] = \left(\frac{1}{2}\right)^n u[n] \quad \Leftrightarrow \quad X_1(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}} \qquad \text{ROC: } |z| > \frac{1}{2}$$



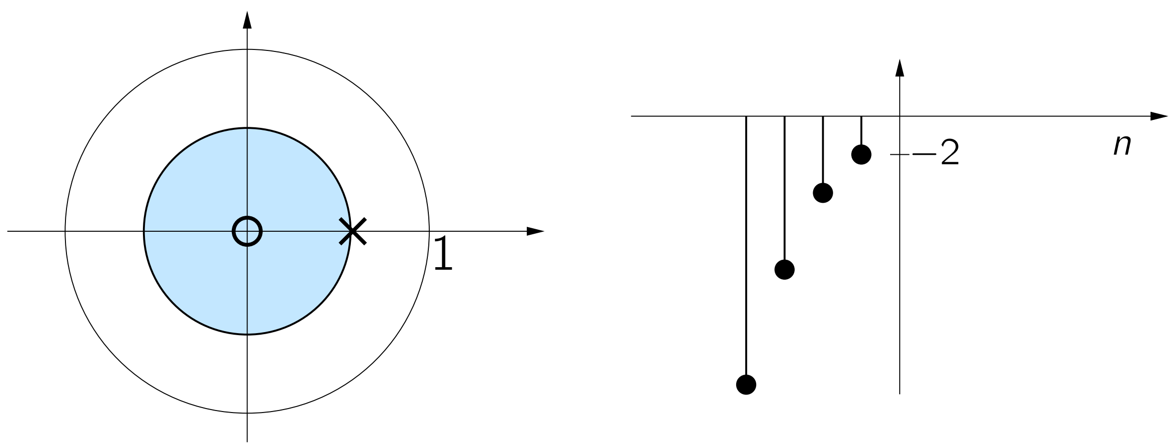
# The $z$-transform

**Example**

- **Anti-causal signal**

$$x_2[n] = -\left(\frac{1}{2}\right)^n u[-n-1] \quad \Leftrightarrow \quad X_2(z) = -\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = -\sum_{m=0}^{\infty}(2z)^m + 1 = \frac{-1}{1-2z} + 1 = \frac{z}{z-\frac{1}{2}}$$

ROC: $|z| < \frac{1}{2}$




The same $X(z)$ corresponds to different $x[n]$ depending on the ROC.

The $z$-transform of $x_1[n]+x_2[n]$ does not exist, because the intersection of the ROCs is empty.

# The $z$-transform

## Example

Compute the $z$-transform of the two-sided signal:

$$x[n] = \left(\frac{1}{2}\right)^{|n|}.$$

- For the causal part we find:

$$x_c[n] = \left(\frac{1}{2}\right)^n u[n] \quad \Leftrightarrow \quad X_c(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1-\frac{1}{2}z^{-1}} = \frac{z}{z-\frac{1}{2}}, \quad \text{ROC: } |z| > \frac{1}{2}$$

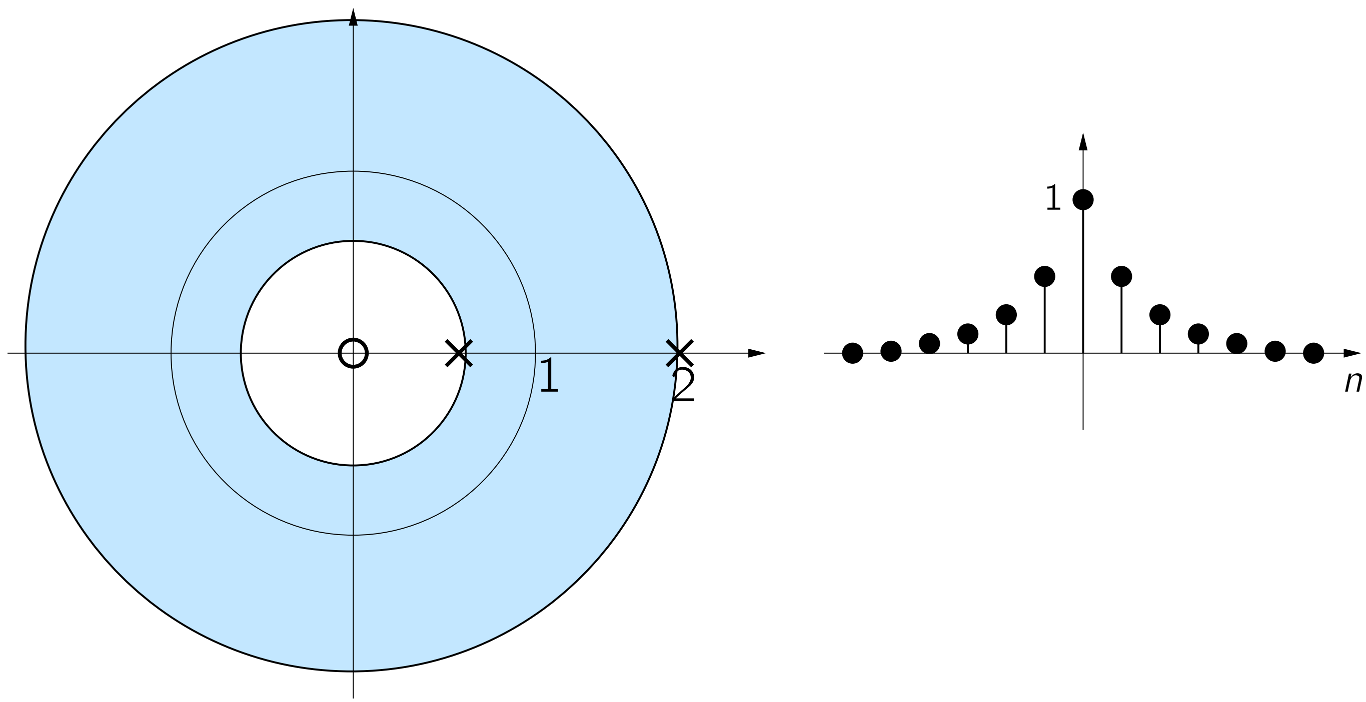
- For the anti-causal part:

$$x_{ac}[n] = \left(\frac{1}{2}\right)^{-n} u[-n] \quad \Leftrightarrow \quad X_{ac}(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^n = \frac{1}{1-\frac{1}{2}z}, \qquad \text{ROC: } |z| < 2$$

- For $x[n]$ we find

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{1}{1-\frac{1}{2}z} - 1 = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-2} = \frac{-1\frac{1}{2}\,z}{(z-\frac{1}{2})(z-2)}$$

with ROC: $\frac{1}{2} < |z| < 2$.



## Region of convergence

With $X(z)$, formally we should always also specify the region of convergence (ROC), indicating where the function is valid. Different sequences $x[n]$ can have the same $X(z)$, but with different ROCs.

Later, it will turn out that we are mostly interested in an ROC which contains the unit circle, because the Fourier Transform is defined here, and in that case we can consider the frequency spectrum of a signal $x[n]$ or system $h[n]$. Convergence on the unit circle also corresponds to BIBO stability.

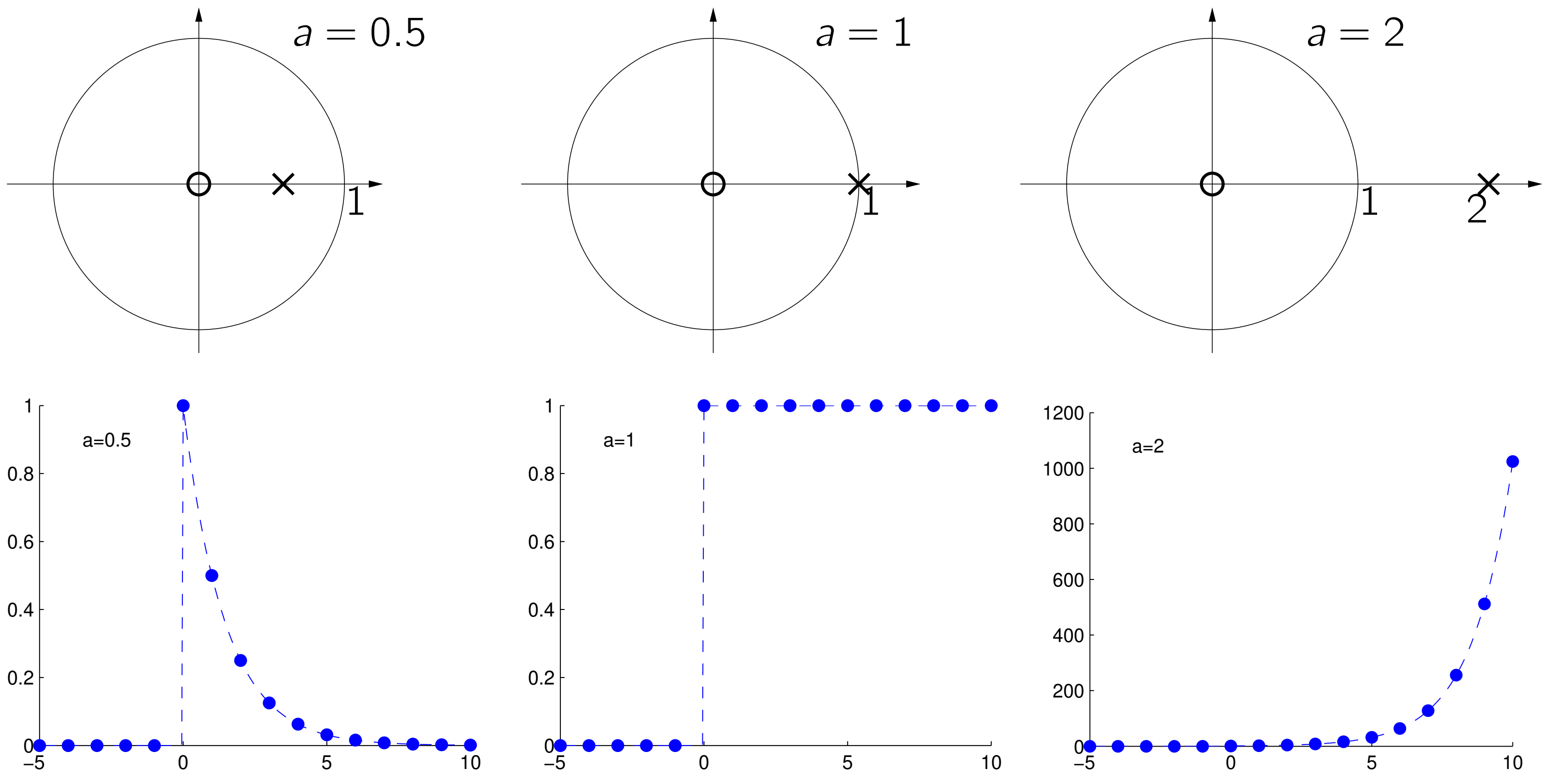
Thus, without further specifications, we will always assume that the ROC contains the unit circle.

# The $z$-transform

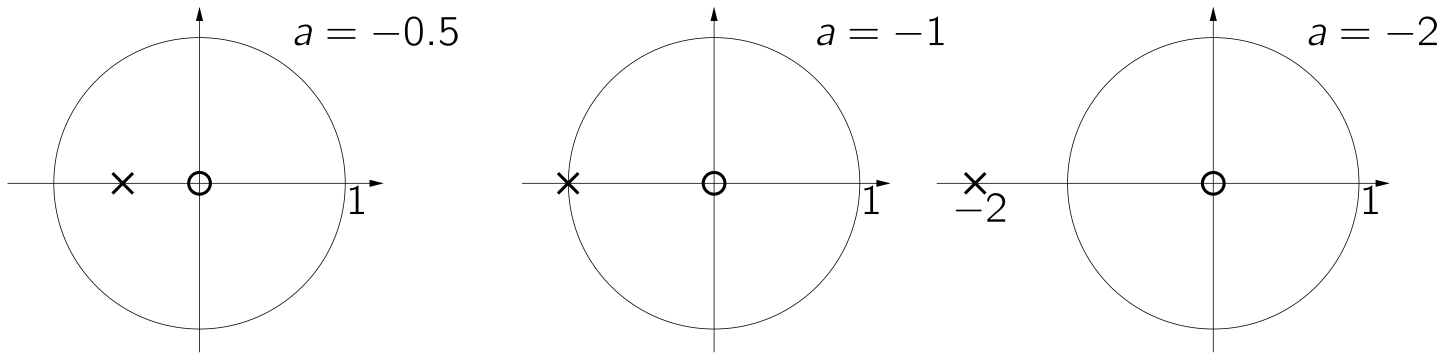
## Exponential signals and corresponding poles

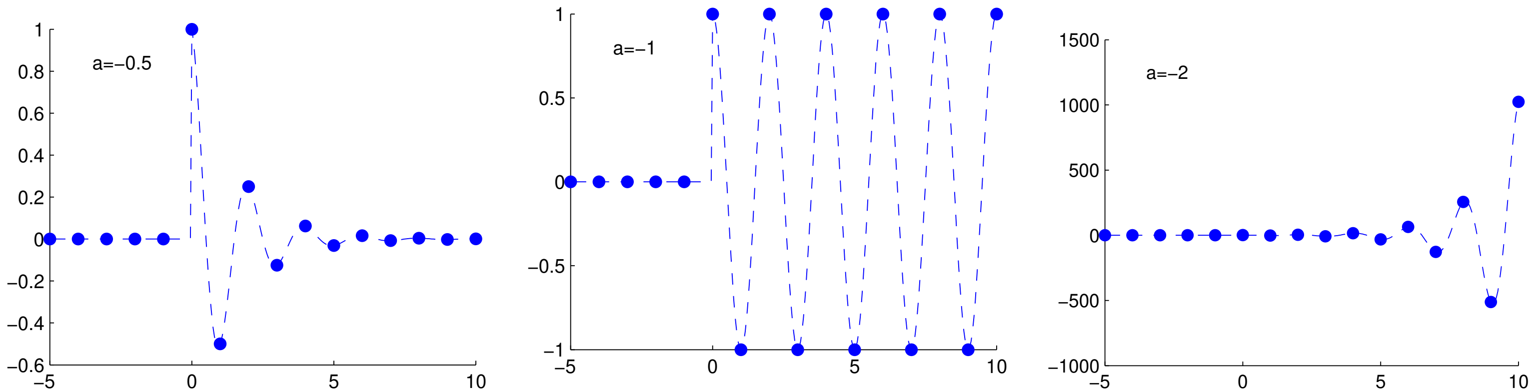
$$x[n] = a^n u[n] \qquad \Leftrightarrow \qquad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad \text{ROC: } |z| > a$$

Pole at $z = a$, zero at $z = 0$.



# The $z$-transform



### Harmonic (exponentially damped) signals and corresponding poles

$$x[n] = r^n \cos(\omega_0 n + \theta) u[n] = \left[ \frac{e^{j\theta}}{2} r^n e^{j\omega_0 n} + \frac{e^{-j\theta}}{2} r^n e^{-j\omega_0 n} \right] u[n] = [\gamma \alpha^n + \gamma^* (\alpha^*)^n]\, u[n]$$

with $\alpha = r e^{j\omega_0}$ and $\gamma = \frac{e^{j\theta}}{2}$ both complex.

$$X(z) = \frac{\gamma z}{z - \alpha} + \frac{\gamma^* z}{z - \alpha^*} = \cdots = \frac{z(z\cos(\theta) - r\cos(\omega_0 - \theta))}{(z - re^{j\omega_0})(z - re^{-j\omega_0})}, \qquad \text{ROC: } |z| > |a|$$

This is a second-order rational function with real-valued coefficients.

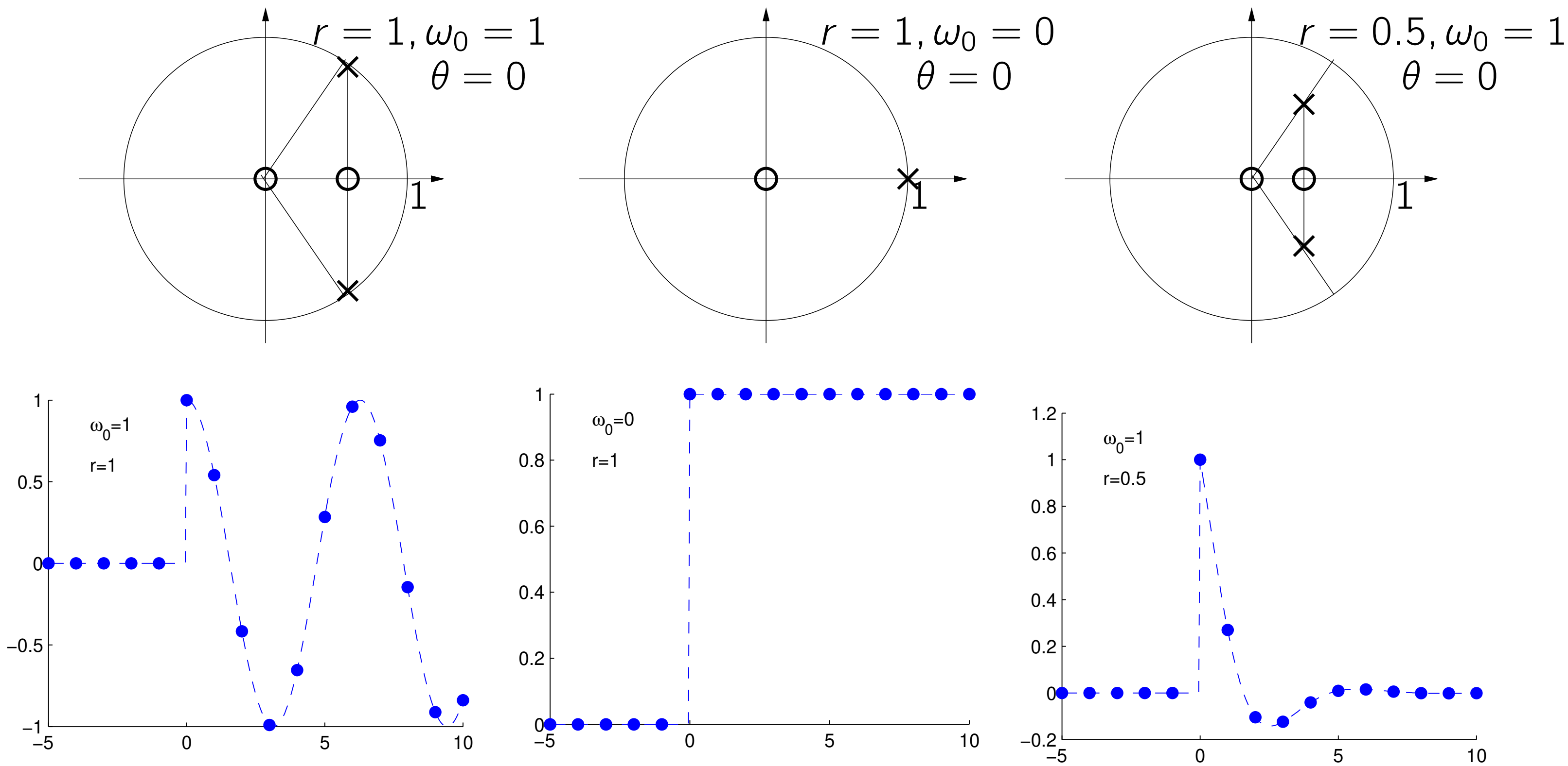
- Poles at $z = re^{j\omega_0}$ and $z = re^{-j\omega_0}$.

  Special case: $r = 1$, now $x[n]$ is an undamped (causal) sinusoid, with its two poles on the unit circle.

- Zeros at $z = 0$ and $z = \dfrac{r\cos(\omega_0 - \theta)}{\cos(\theta)}$.

# The $z$-transform

- $r = 1$, $\omega_0 = 1$, $\theta = 0$
- $r = 1$, $\omega_0 = 0$ (one pole and zero cancel each other)
- $r = 0.5$, $\omega_0 = 1$



### Double poles

For a causal $x[n]$:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

$$\frac{\mathrm{d}X(z)}{\mathrm{d}z} = \sum_{n=0}^{\infty} x[n] \frac{\mathrm{d}z^{-n}}{\mathrm{d}z} = -z^{-1} \sum_{n=0}^{\infty} n x[n] z^{-n}$$

Hence

$$n x[n] u[n] \quad \Leftrightarrow \quad -z \frac{\mathrm{d}X(z)}{\mathrm{d}z}$$

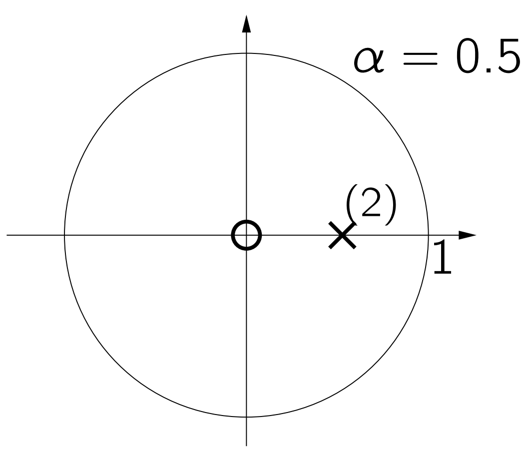
Taking a derivative often leads to double poles.

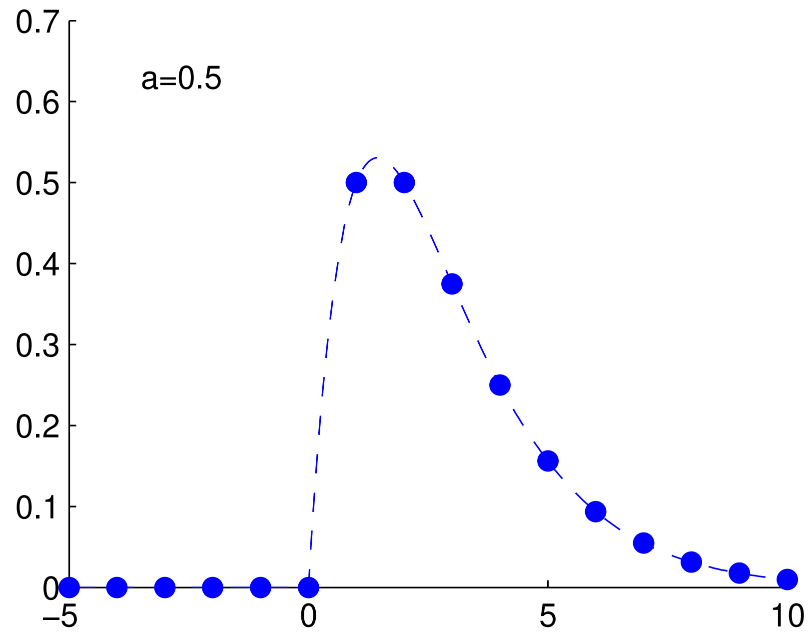
## Example

Taking $x[n] = \alpha^n u[n]$ so that $X(z) = \dfrac{z}{z-\alpha}$, then

$$n\alpha^n u[n] \quad \Leftrightarrow \quad \frac{\alpha z}{(z-\alpha)^2}$$

Double pole at $z = \alpha$, zero at $z = 0$ and $z = \infty$.



# The $z$-transform

## Inverse $z$-transform

Given $X(z)$ and its ROC. The inverse $z$-transform is

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^n \frac{dz}{z}$$

with the contour integral following a counterclockwise closed path in the ROC encircling the origin. This follows from applying the residue theorem to the Laurent series $X(z) = \sum_n x[n] z^{-n}$.

The formula is almost never used except in theoretical derivations.

The integral is solved using the residue theorem.

## The transfer function

We consider an LTI system $\mathcal{T}$ with impulse response $h[n]$, and study exponential signals: $x[n] = z^n$ for $z \in \mathbb{C}$. From the linearity and time invariance:

$$y[n] = \mathcal{T}(z^n) \qquad \Rightarrow \qquad y[n-k] = \mathcal{T}(z^{n-k}) = \mathcal{T}(z^n z^{-k}) = z^{-k}\mathcal{T}(z^n) = z^{-k}y[n]$$

Hence we find the solution $y[n] = \lambda z^n$ with $\lambda \in \mathbb{C}$. We find $\mathcal{T}(z^n) = \lambda z^n$

- The functions $z^n$ are *eigenfunctions* of the system, with eigenvalue $\lambda =: H(z)$.
- The function $H(z)$ is the *transfer function*.

We now consider more general input signals $x[n]$. Using the $z$-transform:

$$x[n] = \frac{1}{2\pi j}\oint X(z) z^n \frac{dz}{z}, \qquad y[n] = \frac{1}{2\pi j}\oint Y(z) z^n \frac{dz}{z}$$

we find

$$y[n] = \mathcal{T}(x[n]) = \frac{1}{2\pi j}\oint X(z) \underbrace{\mathcal{T}(z^n)}_{H(z)z^n} \frac{dz}{z} \qquad \Rightarrow \qquad Y(z) = H(z)X(z)$$

# The $z$-transform

## The transfer function in terms of $h[n]$

- Consider the system $y[n] = \mathcal{T}(x[n]) := x[n-1]$.

  The $z$-transform gives $Y(z) = z^{-1}X(z)$.

  Hence, for a delay we have $H(z) = z^{-1}$.

- For a general LTI system: take $x[n] = \delta[n]$ so that $y[n] = h[n]$. From

$$Y(z) = H(z)X(z) = H(z)\,, \qquad \text{(since } X(z) = 1\text{)}$$

  we find

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- For the lineair difference equation, $\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$, we find, after taking the $z$-transform, the rational transfer function

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \qquad \Leftrightarrow \qquad H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} =: \frac{B(z)}{A(z)}$$

## Computing the convolution

Given $x[n] = [\boxed{1}, 2, 0, \cdots]$ and $h[n] = [\boxed{3}, 2, 4, 0, \cdots]$.

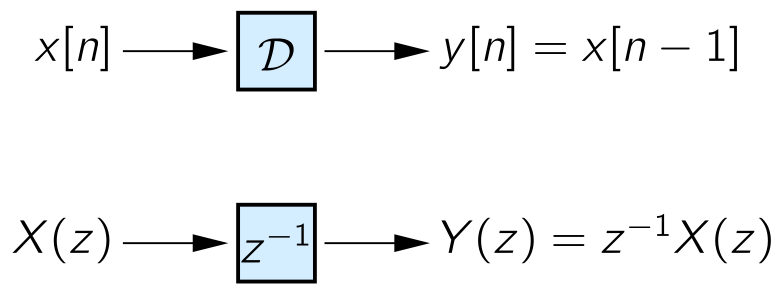
Compute $y[n] = x[n] * h[n] = \sum_{k=0}^{\infty} x[k]h[n-k]$:

| | | | | | |
|---|---|---|---|---|---|
| $x[0]h[n]$ : | 3 | 2 | 4 | 0 | $0 \cdots$ |
| $x[1]h[n-1]$ : | 0 | $2 \cdot 3$ | $2 \cdot 2$ | $2 \cdot 4$ | $0 \cdots$ |
| $y[n]$ : | 3 | 8 | 8 | 8 | $0 \cdots$ |

Now compute using $Y(z) = X(z)H(z)$:

$$
\begin{aligned}
Y(z) &= (1 + 2z^{-1})(3 + 2z^{-1} + 4z^{-2}) \\
&= (3 + 2z^{-1} + 4z^{-2}) + 2z^{-1}(3 + 2z^{-1} + 4z^{-2}) \\
&= 3 + (2 + 2 \cdot 3)z^{-1} + (4 + 2 \cdot 2)z^{-2} + (2 \cdot 4)z^{-3} \\
&= 3 + 8z^{-1} + 8z^{-2} + 8z^{-3}
\end{aligned}
$$

**Realizations (intro)**




- The delay-element is a memory (clocked D-flip-flop): It shows at the output what was the input at the previous clock cycle.

- Block schemes ("realizations") consist of delays, multipliers and adders.

  In block schemes, $\mathcal{D}$ is usually written as $z^{-1}$.

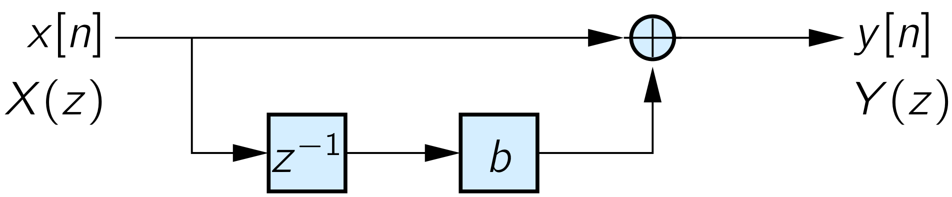
  Therefore, $x[n]$ and $X(z)$ are often interchangeably used in block schemes.

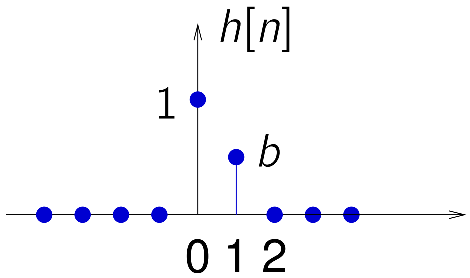
- The impulse response $h[n]$ follows for $n = 1, 2, \cdots$ by inserting an input signal $x[n] = \delta[n]$ into the realization, and recursively computing the signals in the scheme sample by sample (assuming initial conditions of the delays are zero).

## Realizations (intro)

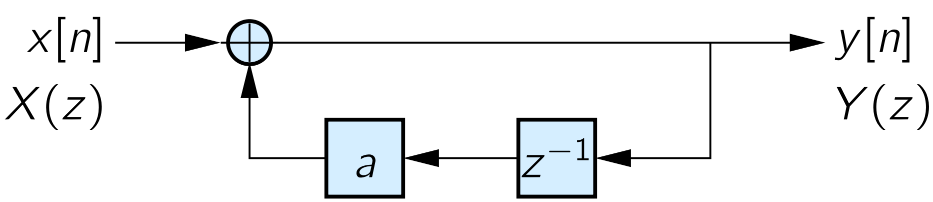
A rational transfer function $H(z)$ corresponds to a realization using delays, multipliers and adders.

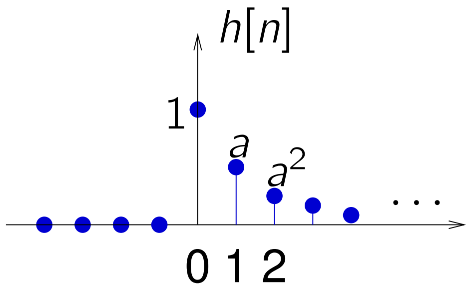
- $H(z) = 1 + bz^{-1} \qquad \Rightarrow \qquad h[n] = \delta[n] + b\delta[n-1]$




- $H(z) = \dfrac{z}{z-a} = \dfrac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots \qquad \Rightarrow \qquad h[n] = a^n u[n]$

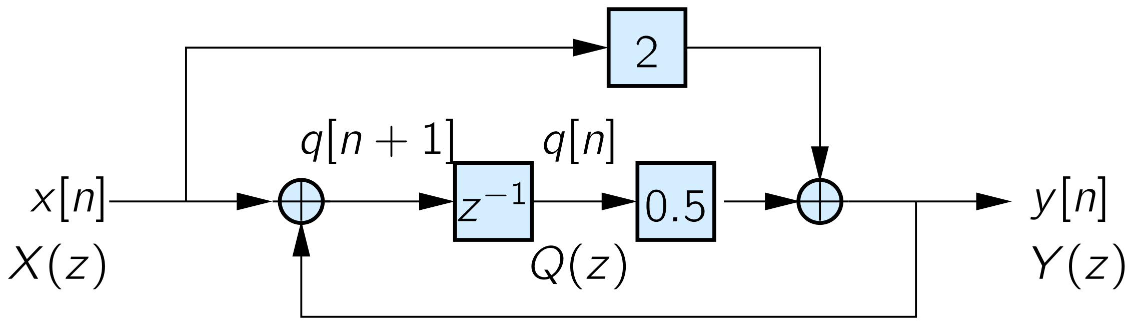



Derivation: $Y(z) = H(z)X(z) \;\Rightarrow\; Y(z)(1-az^{-1}) = X(z) \;\Rightarrow\; Y(z) = X(z) + az^{-1}Y(z)$

## Exercise

Determine the transfer function of the following system:




Answer:

$$\begin{cases} Y(z) & = & 2X(z) + 0.5Q(z) \\ Q(z) & = & z^{-1}Y(z) + z^{-1}X(z) \end{cases}$$

$$Y(z) = 2X(z) + 0.5[z^{-1}Y(z) + z^{-1}X(z)]$$

$$(1 - 0.5z^{-1})Y(z) = (2 + 0.5z^{-1})X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + 0.5z^{-1}}{1 - 0.5z^{-1}} = \frac{2z + 0.5}{z - 0.5}$$

The general technique to do this is to define a new variable as the input or output of every delay (here $q[n]$ or equivalently $Q(z)$).

# The $z$-transform

## Causal system

For a causal LTI system, we have $h[n] = 0, \; n < 0$.

$$H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad = \quad \sum_{n=0}^{\infty} h[n]z^{-n}$$

Hence, an LTI system is causal iff the ROC of $H(z)$ contains the outside of a circle, including $z = \infty$.

## BIBO stable system

Earlier: A system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.

Note:

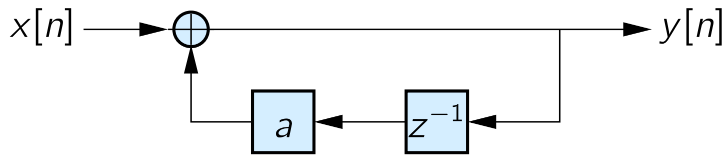
$$|H(z)| \; \leq \; \sum |h[n]z^{-n}| \; = \; \sum |h[n]| \; |z^{-n}|$$

On the unit circle, a BIBO stable system satisfies: $|H(z)| < \infty$: the unit circle is contained in the ROC.

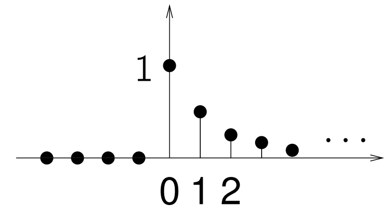
An FIR system is always BIBO stable (finite sum).

# The $z$-transform

## Example

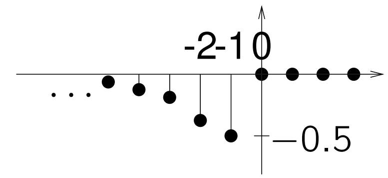



- $a = 0.5 \quad \Rightarrow \quad H(z) = \dfrac{1}{1 - 0.5z^{-1}} = 1 + 0.5z^{-1} + 0.25z^{-2} + \cdots$




- $a = 2 \quad \Rightarrow \quad H(z) = \dfrac{1}{1 - 2z^{-1}} = 1 + 2z^{-1} + 4z^{-2} + \cdots ???$

  The series does not converge for $z$ on the unit circle.




$$H(z) = \frac{1}{1 - 2z^{-1}} = -\frac{0.5z}{1 - 0.5z} = -0.5z - 0.25z^2 - 0.125z^3 - \cdots$$

The series converges for $z$ on the unit circle, and corresponds to a stable but anti-causal system. The series does *not* correspond to the impulse response of the given realization (which is causal but not stable for $a = 2$).

### Conclusions

- A causal stable system has all poles within the unit circle. The ROC contains at least the unit circle and the area outside it.

- Along with $H(z)$, we also must indicate the ROC. Often the ROC is omitted. In that case, the assumption is that the unit circle is within the ROC (the system is stable). The corresponding $h[n]$ is then uniquely determined; it doesn't have to be causal.

  In other cases, the implicit assumption is that the system is causal. In that case, the unit circle does not have to lie within the ROC (e.g., pole outside the unit circle: in that case the system is not stable).

### Initial value and final value

If $x[n]$ is causal, then

Initial value: $x[0] = \lim_{z\to\infty} X(z)$

Final value: $\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)X(z)$, (if ROC $\supset \{|z| \geq 1\}\backslash\{1\}$).

Proof:

- $\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \sum_{n=0}^{\infty} x[n]z^{-n} = x[0]$
-

$$\begin{aligned} \lim_{z\to 1}(z-1)X(z) &= \lim_{z\to 1} x[0]z + \sum_{n=0}^{\infty}(x[n+1]-x[n])z^{-n} \\ &= x[0] + \sum_{n=0}^{\infty}(x[n+1]-x[n]) \\ &= \lim_{n\to\infty} x[n] \end{aligned}$$

The properties can be used to check the correctness of a computed $x[n]$.

# The $z$-transform

Given $X(z)$ for a causal signal (ROC: $|z| > R$), how can $x[n]$ be obtained?

- Use the inverse $z$-transform (contour integral).

$$x[n] = \frac{1}{2\pi j} \oint X(z) z^n \frac{dz}{z}$$

General technique, but often rather complicated... Gives some insight in case the ROC is a band (non-causal signal).

- Expansion into a power series of $z^n$ (by long division)

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \quad \Leftrightarrow \quad x[n] = x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Similar: make a realization for $X(z)$ and determine step-by-step the impulse response. Useful if only a few terms $(x[0], x[1], \cdots)$ are needed.

- Partial fraction expansion, then transforming each term separately (using a table).

# The $z$-transform

## Partial fraction expansion

$$X(z) = \frac{B(z)}{A(z)}$$

The degree of $B(z)$ has to be smaller than that of $A(z)$ ("proper rational function").

If necessary, start by splitting off a polynomial, (in $z^{-1}$), e.g.,

$$X(z) = b_0 + b_1 z^{-1} + \frac{B'(z)}{A(z)}$$

Then determine the poles (i.e. the zeros of $A(z)$). If none of the poles is repeated, then the partial fraction expansion has the form

$$X(z) = b_0 + b_1 z^{-1} + \sum \frac{A_k}{1 - \alpha_k z^{-1}}$$

We find (depending on the ROC... but we assumed that $x[n]$ is causal)

$$x[n] = b_0 \delta[n] + b_1 \delta[n-1] + \sum A_k \alpha_k^n u[n]$$

In the case of double poles, we use

$$X(z) = \frac{a z^{-1}}{(1 - a z^{-1})^2} \quad \Leftrightarrow \quad x[n] = n a^n u[n]$$

**Example**

$$X(z) = \frac{2 + z^{-2}}{1 + 2z^{-1} + z^{-2}}, \qquad \text{ROC: } |z| > 1$$

- (Write as function of $z^{-1}$, already the case here.)
- Make "proper"

$$X(z) = 1 + \frac{1 - 2z^{-1}}{1 + 2z^{-1} + z^{-2}}$$

- The poles are $z = -1$ (twice)

$$X(z) = 1 + \frac{1 - 2z^{-1}}{(1 + z^{-1})^2} = 1 + \frac{A}{1 + z^{-1}} + \frac{Bz^{-1}}{(1 + z^{-1})^2}$$

with $A = 1$ and $B = -3$. Hence

$$x[n] = \delta[n] + (-1)^n u[n] + 3n(-1)^n u[n]$$

- Check:

$x[0] = 2$ and $\lim_{z\to\infty} X(z) = 2$

$\lim_{n\to\infty} x[n]$ does not converge and $(z-1)X(z)$ does not have $z = 1$ in the ROC