EE2S11 Signals and Systems

Chapter 5.7: Basics of Filtering

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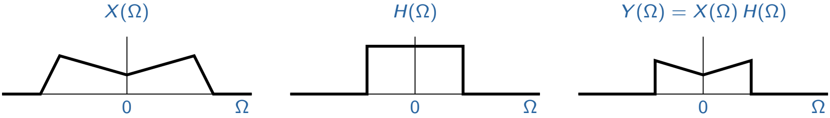
Basics of filtering

Recall

$$y(t) = x(t) * h(t) \qquad \Leftrightarrow \qquad Y(\Omega) = X(\Omega)\, H(\Omega)$$



In signal processing, $x(t)$ is an input signal, $y(t)$ is an output signal, and $H(\Omega)$ the frequency response of a filter.

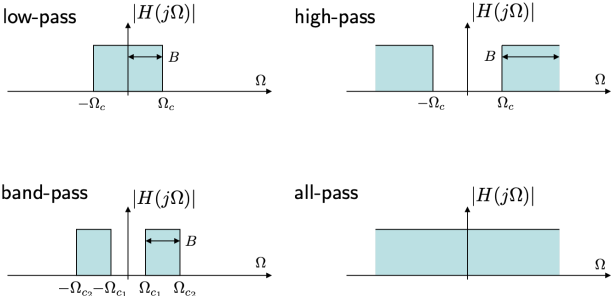
We consider rational filters, with transfer functions as

$$H(s) = \frac{B(s)}{A(s)} = \frac{\text{some polynomial}}{\text{another polynomial}}$$

Frequency selective filters

Filter design: find $H(s)$ to satisfy certain specifications, such as

- Flat frequency response in the pass band
- Sufficient suppression in the stop band

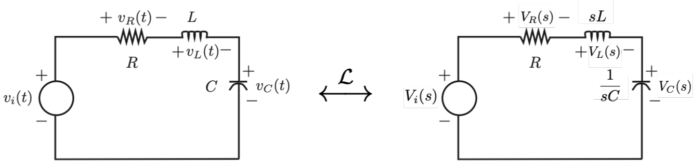


These ideal filters cannot be realized...

Frequency selective filters

Example: second-order RCL circuit

This (passive) circuit can generate many second-order rational filters:



(zero initial conditions)

$$V_i(s) = V_R(s) + V_L(s) + V_C(s)$$

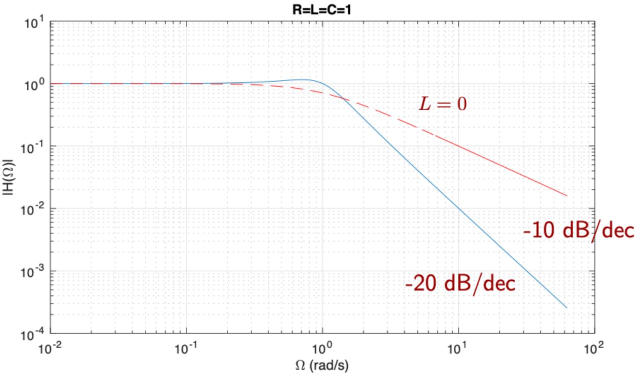
Example Voltage across the capacitor:

$$V_c(s) = \frac{V_i(s)}{s^2LC + sRC + 1} \quad \Rightarrow \quad H(s) = \frac{1}{s^2LC + sRC + 1}$$

Frequency selective filters

The amplitude response is

$$|H(j\Omega)|^2 = \frac{1}{(1-\Omega^2 LC)^2 + (\Omega RC)^2}$$



This is a second-order lowpass filter.

Frequency selective filters

Similarly:

- Voltage across the inductor:

$$H(s) = \frac{s^2 LC}{s^2 LC + sRC + 1}$$

- Voltage across the resistor:

$$H(s) = \frac{sRC}{s^2 LC + sRC + 1}$$

- Voltage across the inductor and capacitor:

$$H(s) = \frac{s^2 LC + 1}{s^2 LC + sRC + 1}$$

What types of filters are these?

Phasors

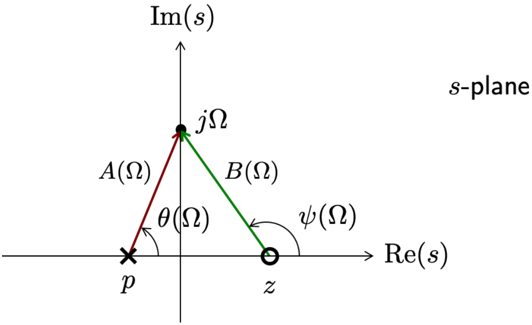
Consider a first-order rational transfer function

$$H(s) = \frac{s - z}{s - p}$$

Frequency response:

$$H(\Omega) = \frac{j\Omega - z}{j\Omega - p} = \underbrace{\frac{|j\Omega - z|}{|j\Omega - p|}}_{|H(j\Omega)|} \underbrace{e^{j(\angle(j\Omega - z) - \angle(j\Omega - p))}}_{e^{j\angle H(j\Omega)}}$$

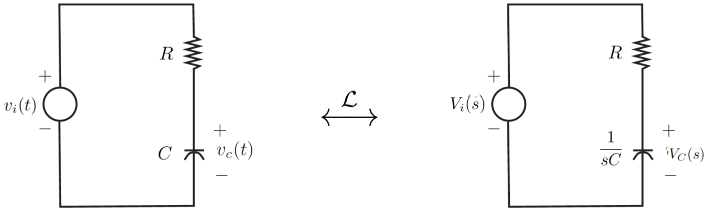
Phasors



$$|H(j\Omega)| = \frac{B(\Omega)}{A(\Omega)} \qquad \angle H(j\Omega) = \psi(\Omega) - \theta(\Omega)$$

Phasors

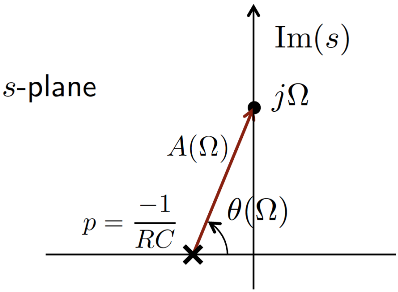
Example: first-order RC circuit

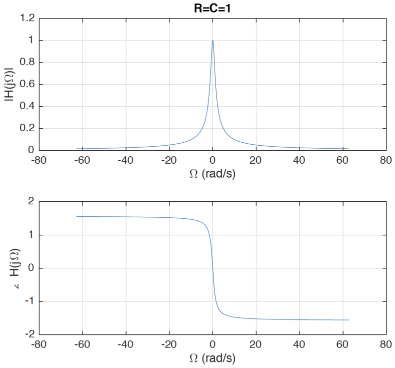


$$H(s) = \frac{V_c(s)}{V_i(s)} = \frac{1}{1 + sRC} = \frac{(RC)^{-1}}{(RC)^{-1} + s}$$

- One zero at ∞
- One pole at $-\frac{1}{RC}$

Phasors





This is a first-order lowpass filter. Close to the pole, the response will peak, and the phase changes quickly. Far away, the response slowly goes to zero.

Phasors

More general:

$$H(s) = \frac{b_0 + b_1 s + \cdots + b_M s^M}{a_0 + a_1 s + \cdots + a_N s^N} = \frac{b_M}{a_N} \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

Write $s = j\Omega$ and express the complex-valued factors in polar form as

$$j\Omega - z_k = B_k(\Omega) e^{j\psi_k(\Omega)}, \qquad j\Omega - p_k = A_k(\Omega) e^{j\theta_k(\Omega)}$$

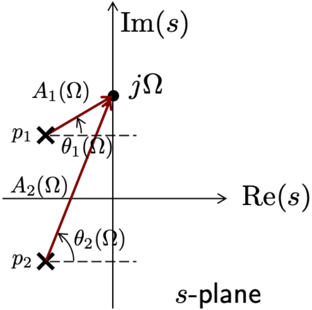
Frequency response:

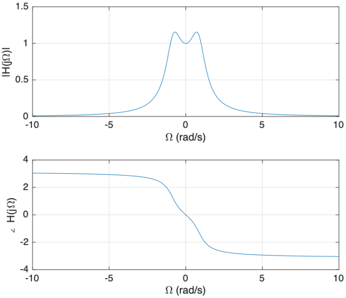
$$\begin{aligned} |H(\Omega)| &= \frac{|b_M|}{|a_N|} \frac{\prod_{k=1}^{M} B_k(\Omega)}{\prod_{k=1}^{N} A_k(\Omega)} \\ \angle H(\Omega) &= \angle \frac{b_M}{a_M} + \sum_{k=1}^{M} \psi_k(\Omega) - \sum_{k=1}^{N} \theta_k(\Omega) \end{aligned}$$

Phasors

Example 1

$$H(s) = \frac{1}{s^2 + s + 1} = \frac{1}{(s - p_1)(s - p_2)}, \qquad p = \frac{1}{2}(-1 \pm j\sqrt{3})$$



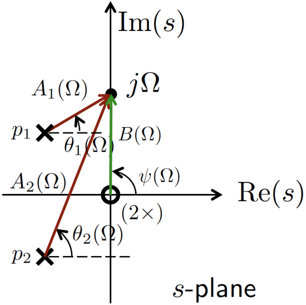


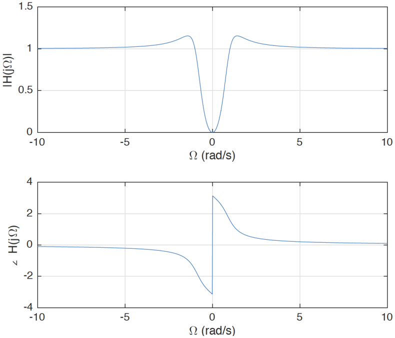
This is a second-order lowpass filter (with a ripple in the passband).

Phasors

Example 2

$$H(s) = \frac{s^2}{s^2 + s + 1}$$



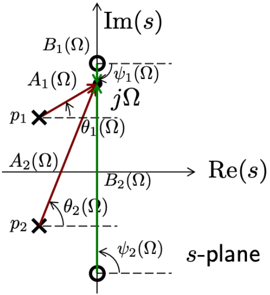


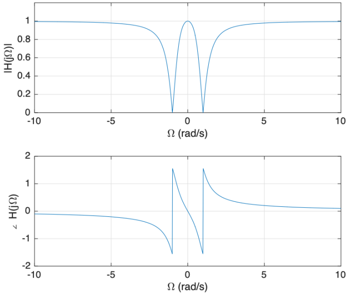
This is a second-order highpass filter. The zeros pull down the response.

Phasors

Example 3

$$H(s) = \frac{s^2+1}{s^2+s+1} = \frac{(s+j)(s-j)}{(s+p_1)(s+p_2)}, \qquad p = \frac{1}{2}(1 \pm j\sqrt{3})$$





This is a second-order bandstop filter. Zeros on the imaginary axis.

Summary

- Phasors are used to construct a magnitude/phase spectrum from a given pole-zero plot.
- Close to poles, the magnitude peaks; close to zeros the magnitude drops
- This allows to judge the type of filter (lowpass, highpass, bandpass, $\cdots$)

Filter design from given specifications comes later in the course (Chapter 7.3).