EE2S11
Exercises Ch.8
Sampling and reconstruction

Problem 8.2 (2nd edition...)

8.2 Consider the sampling of a sinc signal and related signals.

- **(a)** For the signal $x(t) = \sin(t)/t$, find its magnitude spectrum $|X(\Omega)|$ and determine if this signal is band-limited or not.
- **(b)** What would be the sampling period T_s you would use for sampling $x(t)$ without aliasing?
- **(c)** For a signal $y(t) = x^2(t)$ what sampling frequency f_s would you use to sample it without aliasing? How does this frequency relate to the sampling frequency used to sample $x(t)$?

8.2 (a) To find $X(\Omega)$, we use duality or find the inverse Fourier transform of a pulse of amplitude A and bandwidth Ω_0, that is

$$X(\Omega) = A[u(\Omega + \Omega_0) - u(\Omega - \Omega_0)]$$

so that

$$\begin{aligned} x(t) &= \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} Ae^{j\Omega t}d\Omega = \frac{A}{2\pi jt}e^{j\Omega t}\Big|_{-\Omega_0}^{\Omega_0} \\ &= \frac{A}{\pi t}\sin(\Omega_0 t) \end{aligned}$$

which when compared with the given $x(t) = \sin(t)/t$ gives that $A = \pi$ and $\Omega_0 = 1$ or

$$X(\Omega) = \pi[u(\Omega + 1) - u(\Omega - 1)]$$

indicating that $x(t)$ is band-limited with a maximum frequency $\Omega_{max} = 1$ (rad/sec).

Problem 8.2 (cont'd)

(b) To sample without aliasing the sampling frequency should be chosen to be

$$f_s = \frac{1}{T_s} \geq 2\frac{\Omega_{max}}{2\pi}$$

which gives a sampling period

$$T_s \leq \frac{\pi}{\Omega_{max}} = \pi \ \text{sec/sample}$$

(c) The spectrum of $y(t) = x^2(t)$ is the convolution in the frequency

$$Y(\Omega) = \frac{1}{2\pi}\left(X(\Omega) * X(\Omega)\right)$$

(Property omitted in 3rd edition)

which would have a maximum frequency $\Omega_{max} = 2$, giving a sampling frequency which is double the one for $x(t)$. The sampling period for $y(t)$ should be

$$T_s \leq \frac{\pi}{2}.$$

(d) The signal $x(t) = \sin(t)/t$ is zero whenever $t = \pm k\pi$, for $k = 1, 2, \cdots$ so that choosing $T_s = \pi$ (the Nyquist sampling period) we obtain the desired signal $x_s(0) = 1$ and $x(nT_s) = 0$.

Problem 8.3 (2nd ed) = 8.2 (3rd ed)

8.3 Consider the signal $x(t) = 2\sin(0.5t)/t$

(a) Is $x(t)$ band-limited? If so, indicate its maximum frequency Ω_{max}.
(b) Suppose that $T_s = 2\pi$, how does Ω_s relate to the Nyquist frequency $2\Omega_{max}$? Explain. What is the sampled signal $x(nT_s)$ equal to?
(c) Determine the spectrum of the sampled signal $X_s(\Omega)$ when $T_s = 2\pi$ and indicate how to reconstruct the original signal from the sampled signal.

8.3 (a) The Fourier transform is

$$X(\Omega) = 2\pi[u(\Omega + 0.5) - u(\Omega - 0.5)]$$

$x(t)$ is clearly band-limited with $\Omega_{max} = 0.5$ (rad/sec).
(b) According to the Nyquist sampling rate condition, we should have that

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_{max}$$

or the sampling period

$$T_s \leq \frac{\pi}{\Omega_{max}} = 2\pi$$

The given value satisfies the Nyquist sampling rate condition so we can sample the signal with no aliasing. The given sampling period is the Nyquist sampling period.

Problem 8.3 (cont'd)

Plotting the sinc function it can be seen that it is zero at values of $0.5t = \pm\pi k$ or $t = \pm 2\pi k$ for an integer k. The sampled signal using $T_s = 2\pi$ is

$$x(nT_s) = \frac{\sin(0.5\ 2\pi n)}{0.5n\ 2\pi} = \frac{\sin(\pi n)}{\pi n}$$

which is 1 for $n = 0$, and 0 for any other value of n.
(c) It seems the signal cannot be reconstructed from the samples, that frequency aliasing has occurred. Ideally, that is not the case. The spectrum of the sampled signal $x_s(t)$ for $T_s = 2\pi$ ($\Omega_s = 1$) is

$$X_s(\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(\Omega - k\Omega_s) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(\Omega - k) = 1$$

Passing this signal through an ideal low-pass filter with amplitude 2π and cut-off frequency $\Omega_s/2 = 1/2$ the reconstructed signal, the output of this filter, is the inverse Fourier transform of a pulse in frequency, i.e., a sinc function, that coincides with the original signal.

Problem 8.5 (2nd ed) = 8.3 (3rd ed)

8.5 The signal $x(t)$ has a Fourier transform $X(\Omega) = u(\Omega + 1) - u(\Omega - 1)$ thus it is band-limited, suppose we generate a new signal $y(t) = (x * x)(t)$, i.e., it is the convolution of $x(t)$ with itself.

(a) Find $x(t)$ and indicate its support.
(b) Is it true that $y(t) = x(t)$? to determine if this is so calculate $Y(\Omega)$.
(c) What would be the largest value of the sampling period T_s that would not cause aliasing when sampling $y(t)$?
(d) What happens if we choose $T_s = \pi$? how does this relate to the Nyquist condition?

8.5 (a) Inverse Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-1}^{1} e^{j\Omega t}d\Omega = \frac{\sin(t)}{\pi t}$$

of infinite support in time but finite support in frequency.

(b) $y(t) = (x * x)(t)$ then $Y(\Omega) = X(\Omega)X(\Omega) = X(\Omega)$ so that $y(t) = x(t)$

(c) $2\pi f_{max} = 1$ then $f_s \geq 2f_{max} = 1/\pi$ so $T_s \leq \pi$.

(d) If $T_s = \pi$ we get samples

$$x(nT_s) = \begin{cases} 1/\pi & n = 0 \\ \sin(n\pi)/(n\pi^2) = 0 & n \neq 0 \end{cases}$$

$T_s = \pi$ barely satisfies Nyquist.

Problem 8.8 (2nd ed) = 8.4 (3rd ed)

8.8 A message $m(t)$ with a bandwidth of $B = 2$ kHz modulates a cosine carrier of frequency 10 kHz to obtain a modulated signal $s(t) = m(t)\cos(20 \times 10^3 \pi t)$.

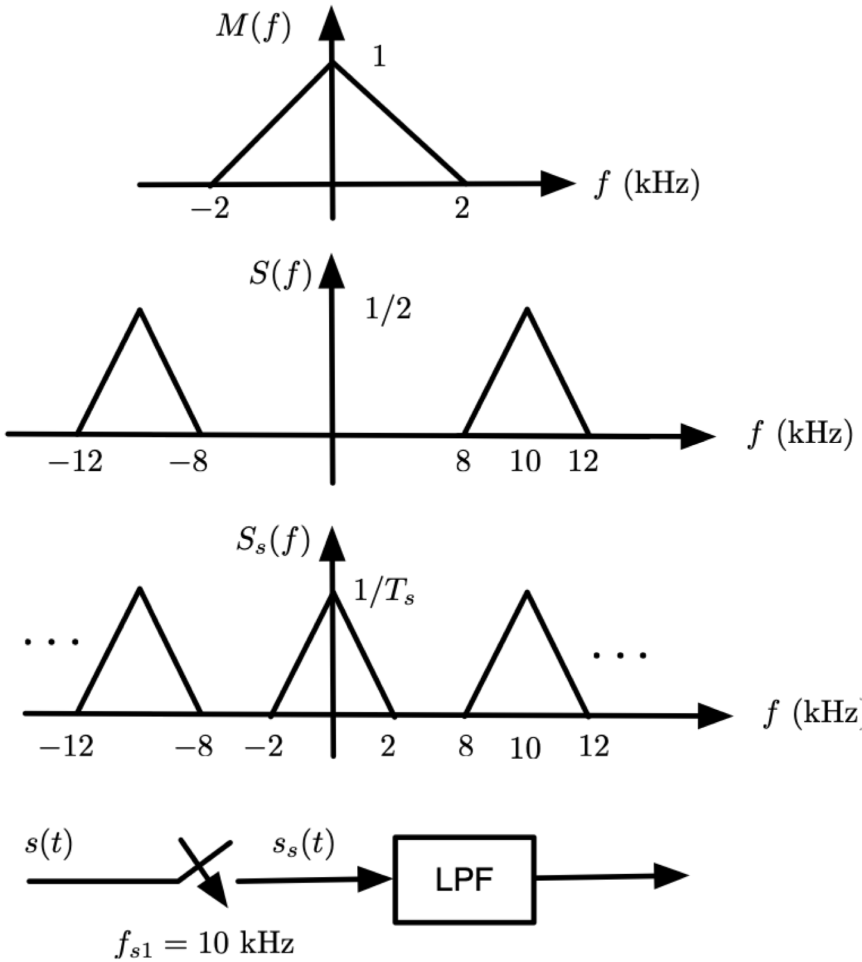
(a) What is the maximum frequency of $s(t)$? What would be the values of the sampling frequency f_s in Hz, according to the Nyquist sampling condition, that could be used to sample $s(t)$?

(b) Assume the spectrum of $m(t)$ is a triangle, with maximum amplitude 1, carefully plot the spectrum of $s(t)$. Would it be possible to sample $s(t)$ with a sampling frequency $f_s = 10$ kHz and recover the original message? Obtain the spectrum of the sampled signal and show how you would recover $m(t)$, if possible.

8.8 (a) The maximum frequency of $s(t)$ is $f_{max} = 12 \times 10^3$ Hz thus according to Nyquist $f_s \geq 2f_{max} = 24 \times 10^3$ Hz.

(b) The different spectra are shown in Fig. 8.1. By the modulation property the sampler shifts the spectrum of $s(t)$ up and down to center frequencies $10m$ (in kHz) for $m = 0, \pm 1, \pm 2, \cdots$ giving the spectrum $S_s(f)$ in Fig. 8.1. If we filter the sampled signal with a low-pass filter of magnitude $T_s = 1/f_s = 10^{-4}$ and bandwidth 2 kHz we recover the original message.

Problem 8.8 (cont'd)



Problem 8.10 (2nd ed) = 8.5 (3rd ed)

8.10 You wish to recover the original analog signal $x(t)$ from its sampled form $x(nT_s)$.

(a) If the sampling period is chosen to be $T_s = 1$ so that the Nyquist sampling rate condition is satisfied, determine the magnitude and cut-off frequency of an ideal low-pass filter $H(j\Omega)$ to recover the original signal, plot them.

(b) What would be a possible maximum frequency of the signal? Consider an ideal and a non-ideal low-pass filters. Explain.

8.10 (a) If $T_s = 1$ then

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_{max}$$

or $\Omega_{max} \leq \pi$. To reconstruct the original signal we choose the cutoff frequency of the ideal low-pass filter to be

$$\Omega_{max} < \Omega_c < 2\pi - \Omega_{max}$$

and the magnitude $T_s = 1$.

(b) Since $T_s \leq \pi/\Omega_{max}$, and $T_s = 1$ then $\Omega_{max} \leq \pi$. If $\Omega_{max} = \pi$ for an ideal low-pass filter, then $\Omega_c = \pi$ to recover the original signal. Thus the maximum frequency has to be smaller than π to make it possible to use an ideal or a non-ideal low-pass filter to recover the original signal.