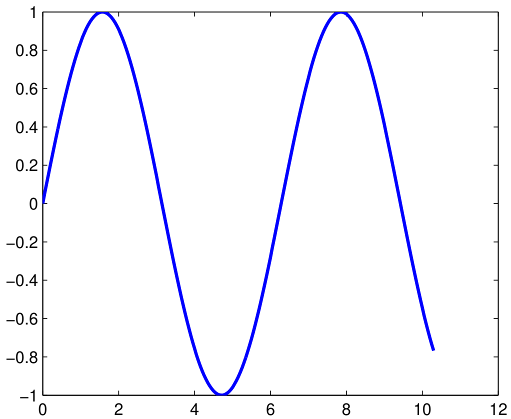
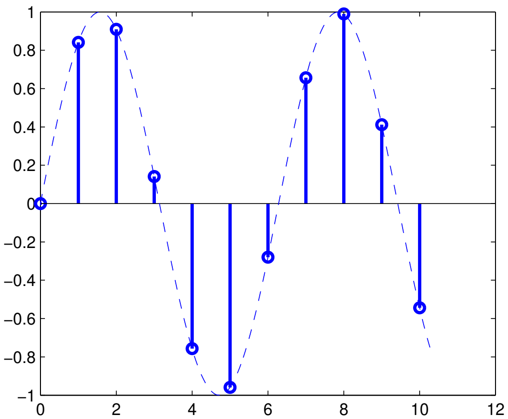
# EE2S11 Signals and Systems, part 2

## Part 2 – Discrete time signals and systems



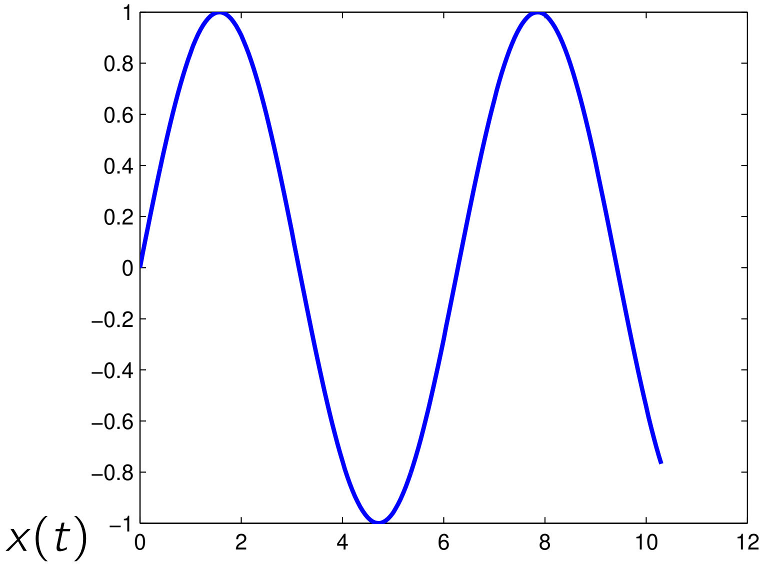



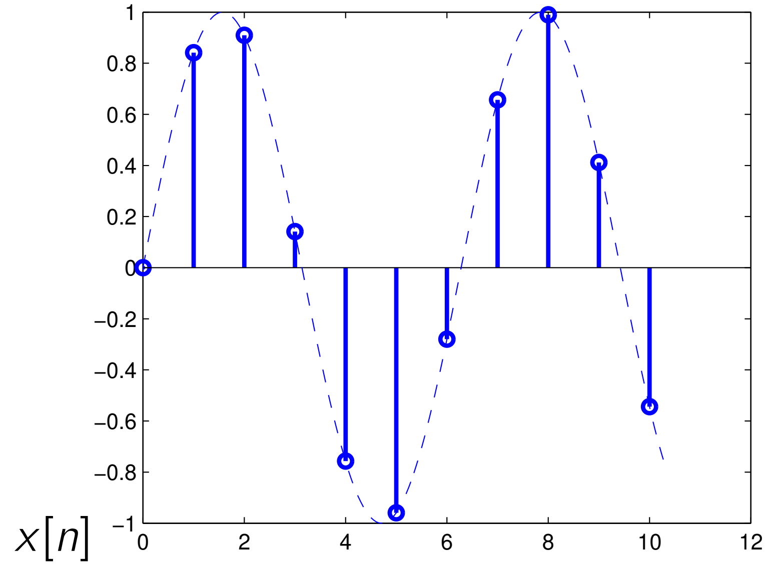

## Program:

- Week 5: Sampling and reconstruction. Discrete-time LTI systems, convolution
- Week 6: $z$-transform. Realisations, canonical filter structures
- Week 7: Discrete-Time Fourier Transform
- Week 8: Analog and digital filter design
- Week 9: Exercises

Today: *Chapter 8 Sampling Theory* – until Sec. 8.2 (rest is covered in EE2S31).

# Sampling Theory




Starting from a given signal $x(t)$ in "analog" time domain, we take samples

$$x[n] = x(nT_s)\,, \qquad n = \cdots, -1, 0, 1, 2, \cdots$$

$T_s$ is called the sampling period. Sampling usually leads to loss of information.

- What does the sequence $\{x[n]\}$ tell about $x(t)$? Can we recover $x(t)$?
- What is a suitable definition for the "spectrum" of $x[n]$?
- How is that related to the spectrum of $x(t)$?

## Sampling

- Given an analog signal $x(t)$. After uniform sampling with period $T_s$, we obtain

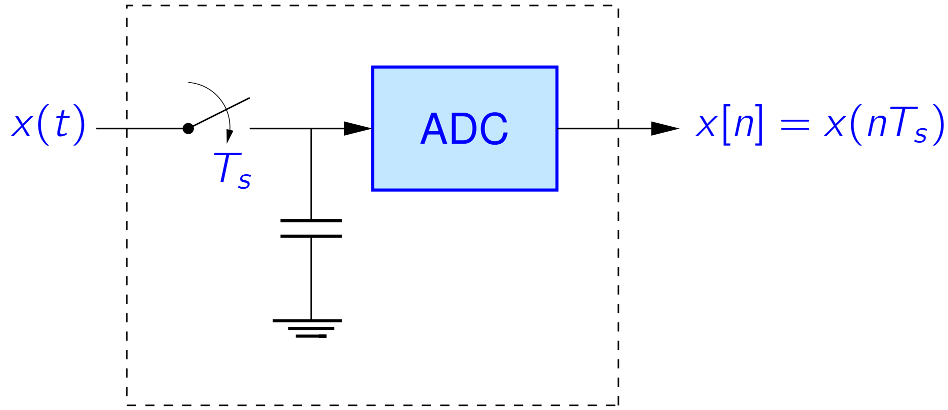
$$x[n] = x(nT_s)\,, \qquad n = \cdots, -1, 0, 1, 2, \cdots$$

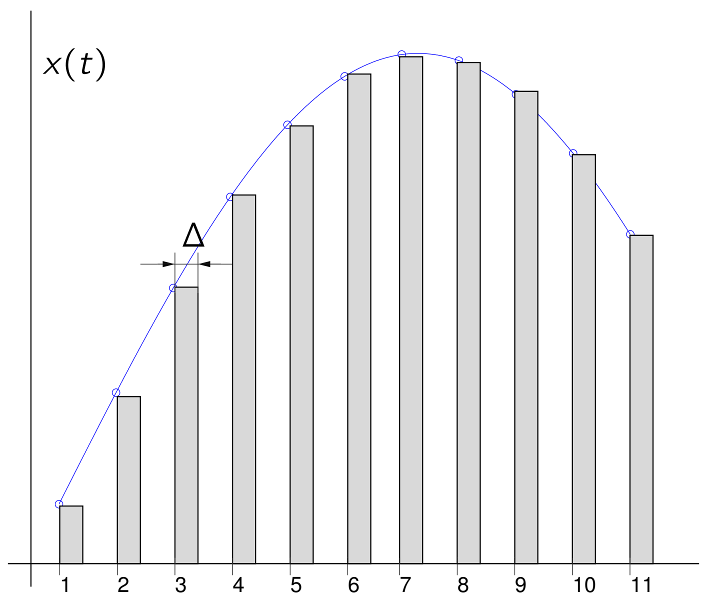
We define

| | | |
|---|---|---|
| $T_s$ | [s] | Sampling period |
| $F_s = 1/T_s$ | [Hz] | Sampling frequency |
| $\Omega_s = 2\pi F_s = 2\pi/T_s$ | [rad/s] | Sampling angular frequency |

# Sampling Theory

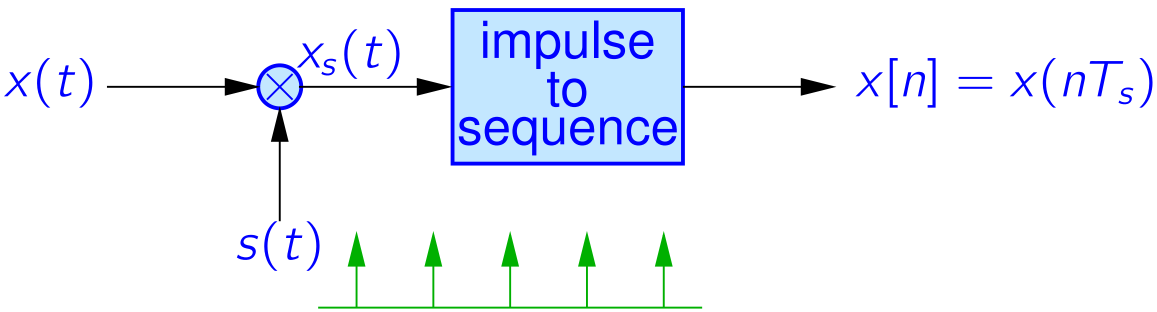
Sampling is done using a switch, a memory element that keeps the voltage for a while, and an analog-to-digital converter (ADC) which measures the amplitude and quantizes it into a number of bits.

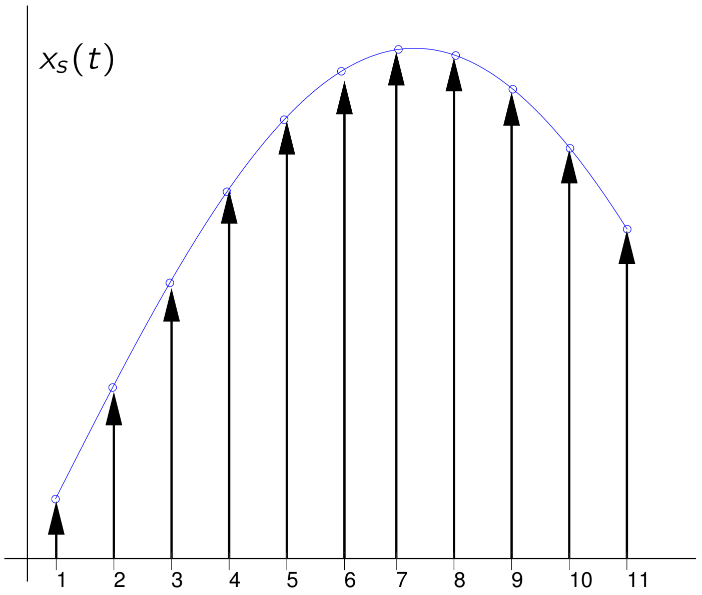


### Ideal sampling

If $\Delta \to 0$, we obtain "ideal sampling", a model in which the sampled signal is represented by a train of delta pulses. (The effect of quantization is postponed to EE2S31.)



# Sampling Theory

## Sampling in time domain

We represent sampling as pointwise multiplication of $x(t)$ by a "delta impulse train" $s(t)$,

$$s(t) = \sum_n \delta(t - nT_s)$$

$s(t)$ is also called a "delta comb" or "sha"-function. The book writes $\delta_{T_s}(t)$.

We obtain

$$x_s(t) = x(t)\,s(t) \;=\; \sum_n x(t)\delta(t - nT_s) \;=\; \sum_n x(nT_s)\delta(t - nT_s) \;=\; \sum_n x[n]\delta(t - nT_s)$$

$x_s(t)$ and $x[n]$ have a one-to-one relation (contain the same information; later we will define the Discrete-Time Fourier Transform such that the spectra are identical).

## Effect of sampling in the frequency domain

($i$) *Modulation:* The impulse train $s(t)$ is periodic with $\Omega_s = 2\pi/T_s$. Write the Fourier Series:

$$s(t) = \sum_{k=-\infty}^{\infty} D_k e^{jk\Omega_s t}$$

The Fourier Series coefficients are given by:

$$D_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} s(t)e^{-jk\Omega_s t}dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jk\Omega_s t}dt = \frac{1}{T_s}e^{-j\cdot 0}\cdot 1 = \frac{1}{T_s}$$

Hence

$$s(t) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} e^{jk\Omega_s t}$$

The impulse train can be written as an infinite sum of sinusoids (all of equal magnitude).

Compute the Fourier Transform and apply the “frequency shift” property:

$$x_s(t) = x(t)s(t) = \frac{1}{T_s}\sum_k x(t)e^{jk\Omega_s t} \quad \leftrightarrow \quad X_s(\Omega) = \frac{1}{T_s}\sum_k X(\Omega - k\Omega_s)$$

### Effect of Sampling in the frequency domain (cont'd)

$(ii)$ *Alternatively:* Use the FT of a shifted delta, and linearity,

$$x_s(t) = \sum_n x(nT_s)\delta(t - nT_s) \quad \leftrightarrow \quad X_s(\Omega) = \sum_n x(nT_s) e^{-j\Omega T_s n}$$

### Interpretation

This motives the definition of the Discrete-Time Fourier Transform which we will see later. Let $\omega = \Omega T_s$, then

DTFT: $$X(\omega) := \sum_n x[n] e^{-j\omega n}$$

**Further development of $(i)$**

Use table 5.1 of Chapparo (p.382):

$$s(t) = \sum_k \delta(t - kT_s) = \frac{1}{T_s} \sum_k e^{jk\Omega_s t} \quad \leftrightarrow \quad S(\Omega) = \frac{2\pi}{T_s} \sum_k \delta(\Omega - k\Omega_s)$$

$s(t)$ is an impulse train in time domain, and corresponds to $S(\Omega)$: an impulse train in frequency domain.

$$x_s(t) = x(t)s(t) \quad \leftrightarrow \quad X_s(\Omega) = \frac{1}{2\pi} X(\Omega) * S(\Omega) = \frac{1}{T_s} \sum_k X(\Omega - k\Omega_s)$$

Modulation of $x(t)$ with the impulse train $s(t)$ corresponds to a convolution of the spectrum $X(\Omega)$ with an impulse train $S(\Omega)$.

This results in a sum of shifted spectra. The shift is with multiples of $\Omega_s$.

**Interpretation of $(i)$**

$$X_s(\Omega) = \frac{1}{T_s} \sum_k X(\Omega - k\Omega_s)$$

Or: the spectrum of the sampled signal, $X_s(\Omega)$, is a sum of shifted spectra of the original signal, $X(\Omega)$.

- Hence $X_s(\Omega)$ is periodic, with period $\Omega_s$.

  It is sufficient to know only one period of $X_s(\Omega)$, e.g. the interval $-\frac{1}{2}\Omega_s < \Omega < \frac{1}{2}\Omega_s$.

  This interval is also called the *fundamental interval*. $\Omega_s/2$ is the *folding frequency*.

- Sampling is not an LTI operator (because it is time varying), hence will result in harmonic frequencies (i.e., frequencies which the original signal did not have – this can never happen with an LTI operator)

- The summation of shifted spectra can lead to *aliasing*.

## Aliasing

- E.g.: $x(t) = e^{j2\pi F_0 t}$, where $F_0$ is the analog frequency (Hz). After sampling:

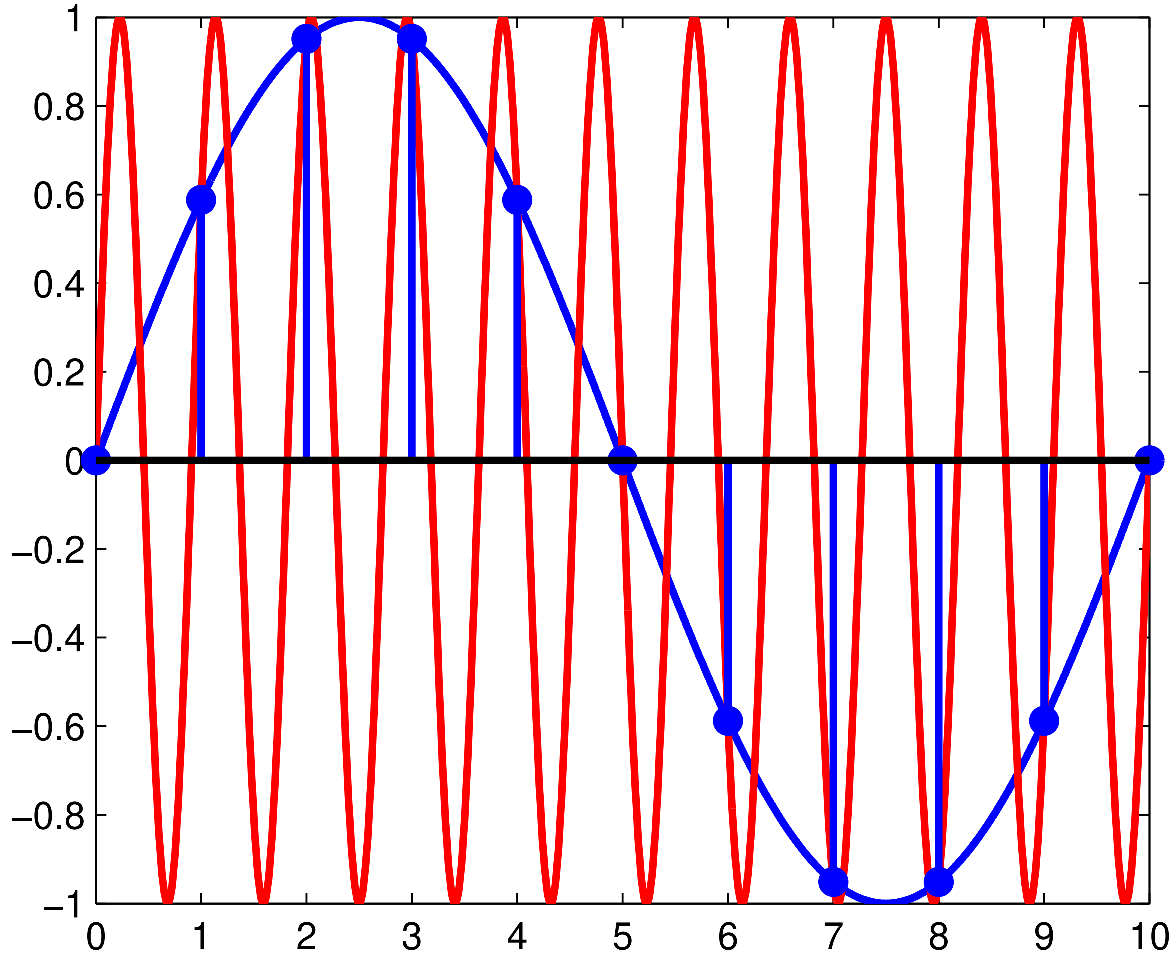
$$x[n] = e^{j2\pi F_0 T_s n} = e^{j2\pi f_0 n} = e^{j\omega_0 n}$$

In here, $T_s$ is the sample period, or $F_s = 1/T_s$ is the sample frequency (in Hz). Further, $\Omega = 2\pi F$.

- $\omega_0 = 2\pi F_0 T_s = 2\pi \dfrac{F_0}{F_s} = 2\pi f_0$ is the “normalised” angular frequency [rad].
- $f_0 = \dfrac{\omega_0}{2\pi} = \dfrac{F_0}{F_s}$ is the “normalised” frequency (no dimension).
- In general: if we sample with frequency $F_s$, then the frequency interval $-\frac{1}{2}F_s \leq F \leq \frac{1}{2}F_s$ is mapped to $-\pi \leq \omega \leq \pi$, or equivalently $-\frac{1}{2} \leq f \leq \frac{1}{2}$.
- Higher frequencies are also mapped onto this interval: aliasing.

  We cannot distinguish between a sinusoid with frequency $F_0$ and one with $F_0 + kF_s$, for $k = \cdots, -2, -1, 0, 1, 2, \cdots$.

## Aliasing




$\omega_1 = 2\pi \cdot 0.1\,, \qquad \omega_2 = 2\pi \cdot 1.1\,, \qquad \omega_k = \omega_1 + 2\pi k \; (k = \pm 1, \pm 2, \cdots)$

### Aliasing

We call a (real-valued) signal $x(t)$ *band limited* if

$$X(\Omega) = 0\,, \qquad |\Omega| > \Omega_{\text{max}}$$

(Real-valued implies that the spectrum of $x(t)$ is symmetric.)
Regarding the sampling of $x(t)$, we have 3 cases:

- $\Omega_s > 2\Omega_{\text{max}}$: this is the *Nyquist sample rate condition*.

  The shifted spectra $X(\Omega - k\Omega_s)$ do not overlap. The fundamental interval only contains a single copy:

  $$X_s(\Omega) = X(\Omega), \qquad \text{for } |\Omega| < \frac{1}{2}\Omega_s$$

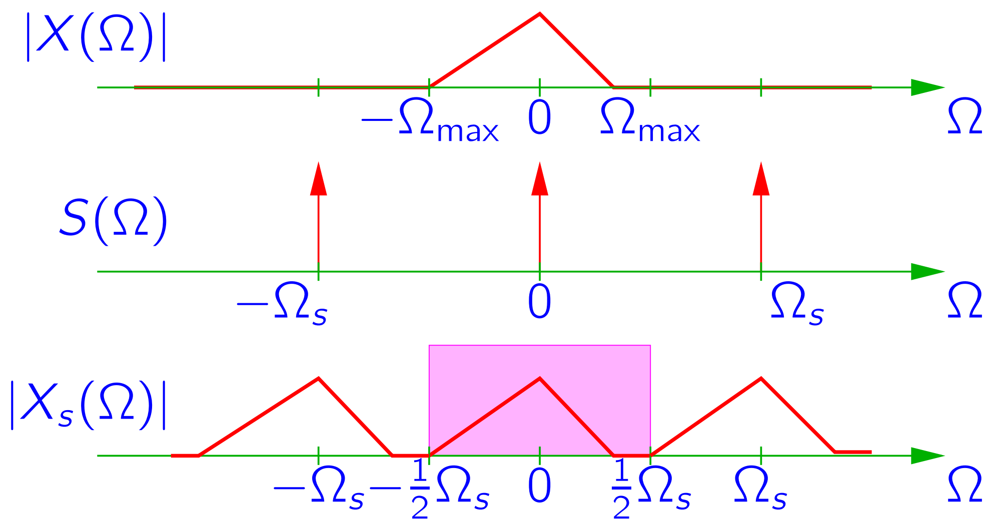
  The aliasing is not destructive.

- The Nyquist condition does not hold. The aliasing is destructive.

- The signal is not band-limited. The Nyquist condition does not hold. Hence the aliasing is destructive.

## Aliasing

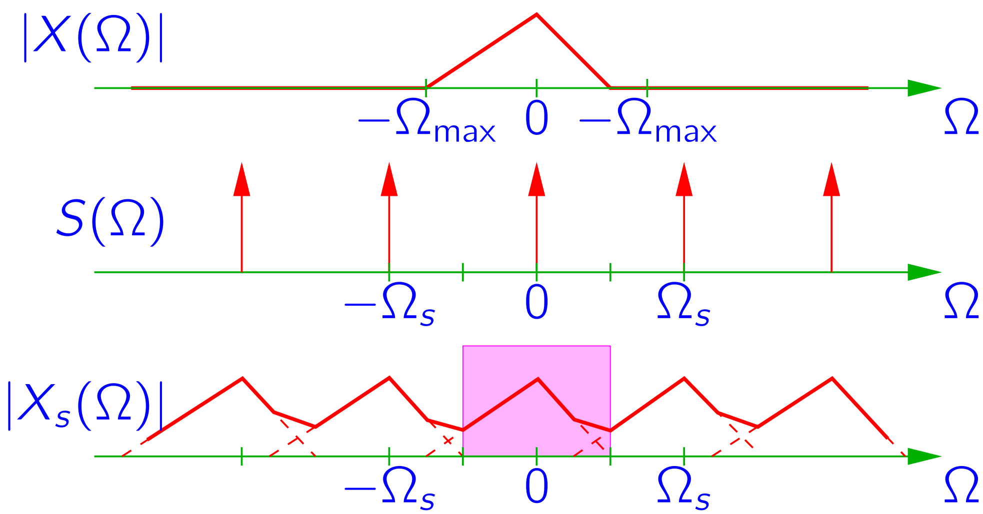
- Example 1: $\Omega_s > 2\Omega_{max}$ (no destructive aliasing)




The pink box indicates the fundamental interval: $[-\frac{1}{2}\Omega_s, \frac{1}{2}\Omega_s]$. Outside this interval, everything is periodic with period $\Omega_s$.

## Aliasing

- Example 2: $\Omega_s < 2\Omega_{max}$ (destructive aliasing)




The part of the spectrum beyond $\pm\frac{1}{2}\Omega_s$ is apparently "folded back".

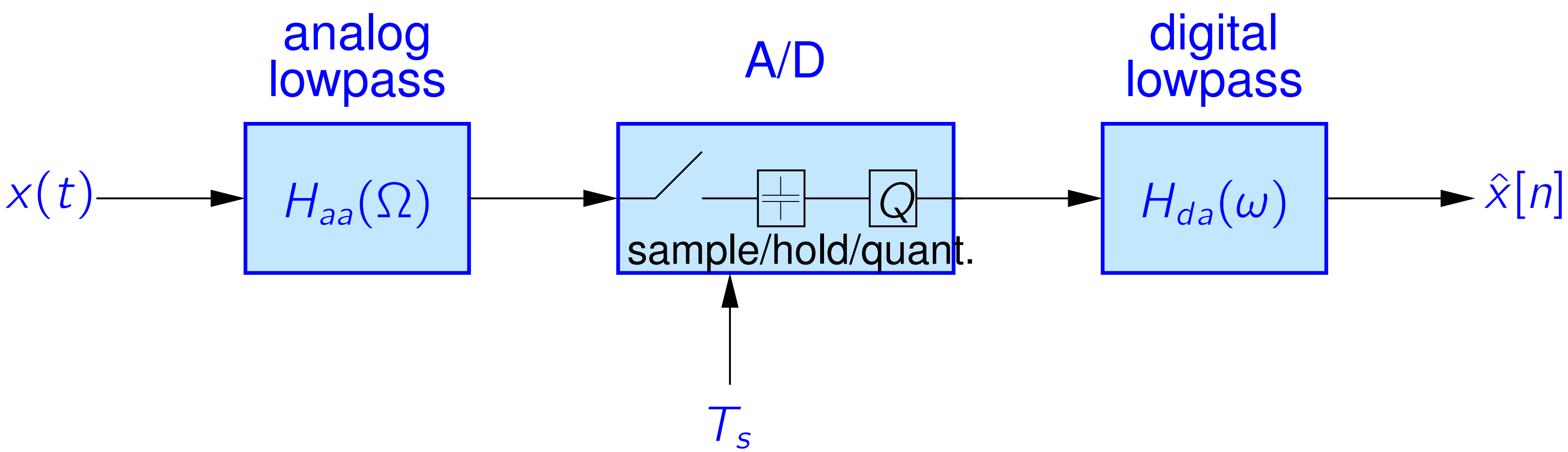
## Aliasing

- Example 3: $x(t)$ not band limited. E.g.,

$$x(t) = u(t+0.5) - u(t-0.5) \quad \leftrightarrow \quad X(\Omega) = \frac{\sin(0.5\Omega)}{0.5\Omega}.$$

To prevent aliasing, practical Analog to Digital Converters (ADCs) employ an anti-aliasing filter which cuts off all frequencies above $\frac{1}{2}\Omega_s$.

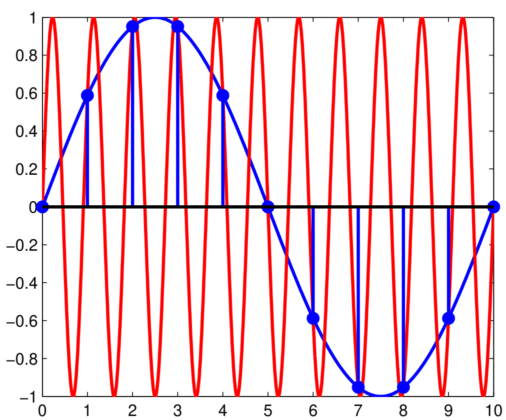
Often, the analog filter is not perfect, and after sampling another (digital) filter is used to correct for this.

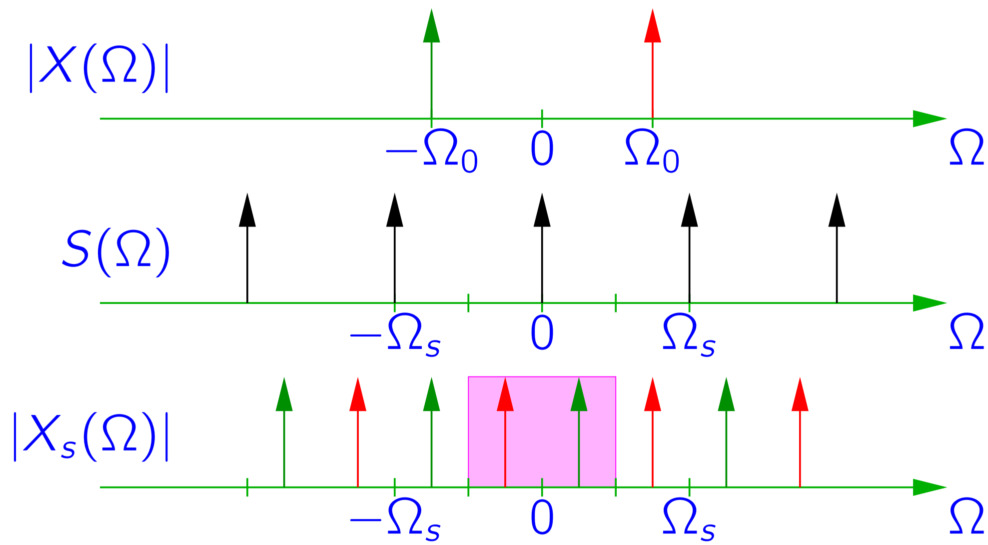


## Aliasing

- Example: $x(t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$ is a sinusoid with $\Omega_0 > \frac{1}{2}\Omega_s$.

$X(\Omega)$ has two components (red/green) that are each shifted with multiples of $\Omega_s$.




(Left figure) The red signal has $\Omega_0 > \frac{1}{2}\Omega_s$, after sampling the spectrum is identical to that of the blue signal for which $\Omega_0 < \frac{1}{2}\Omega_s$.

# Sampling Theory

## Reconstruction

Suppose that $x(t)$ is band limited ($\Omega_{\text{max}}$), and $\Omega_s > 2\Omega_{\text{max}}$ (Nyquist condition: no destructive aliasing).

Can we recover $x(t)$ from its samples $x[n] = x(nT_s)$?

**Shannon's sampling theorem:** A band limited signal can be reconstructed from its samples if the Nyquist condition holds.

# Sampling Theory

## Reconstruction (frequency domain)

- Define an ideal lowpass filter:

$$H_r(\Omega) = \begin{cases} T_s, & |\Omega| \le \Omega_s/2 \\ 0, & |\Omega| > \Omega_s/2 \end{cases} \quad \Leftrightarrow \quad h_r(t) = \frac{T_s}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} e^{j\Omega t}\, d\Omega = \frac{\sin(\pi t/T_s)}{\pi t/T_s} =: \text{sinc}(t/T_s)$$

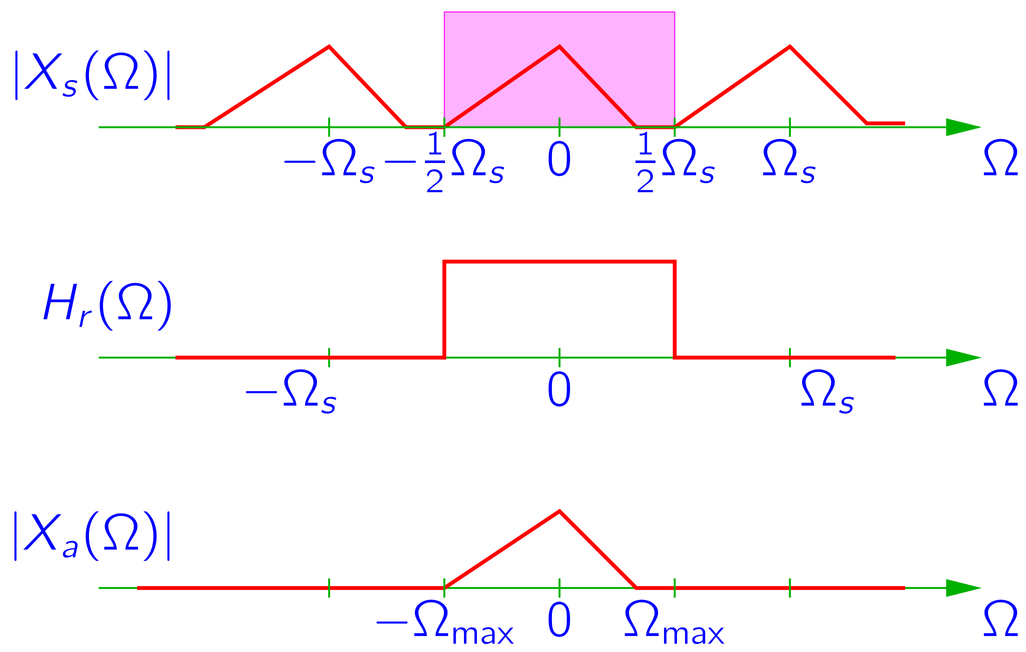
- Apply $H_r(\Omega)$ to $X_s(\Omega)$:

$$X_a(\Omega) = X_s(\Omega)H_r(\Omega) \quad \Leftrightarrow \quad x_a(t) = x_s(t) * h_r(t) = \sum_n x[n]h_r(t - nT_s)$$

$X_a(\Omega) = X(\Omega)$ and hence $x_a(t) = x(t)$: perfect reconstruction.

This is called *ideal band limited signal reconstruction*, and $h_r(t)$ is called the *reconstruction filter* or *interpolation filter*.

## Reconstruction (frequency domain)




The reconstruction filter eliminates the extra copies; the original signal is recovered.

Note that this operation can only be done using filtering in the analog time domain. A sequence of samples $\{x[n]\}$ must first be made analog (a series of impulse spikes).

# Sampling Theory

## Reconstruction (interpretation in time domain)

In time domain, we have $x_a(t) = x_s(t) * h_r(t)$.

Consider the zero crossings of $h_r(t) = \dfrac{\sin(\pi t/T_s)}{\pi t/T_s} = \mathrm{sinc}(t/T_s)$:

$$h_r(kT_s) = \mathrm{sinc}(k) = \begin{cases} 1\,, & k = 0 \quad \text{(use L'Hopital)} \\ 0\,, & k = \pm 1, \pm 2, \cdots \end{cases}$$

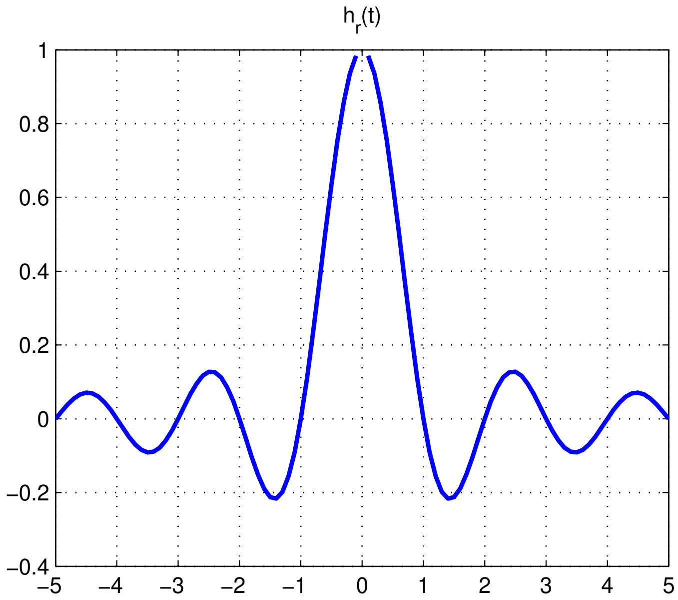
Using $x_a(t) = \sum_n x[n] h_r(t - nT_s)$, it follows that

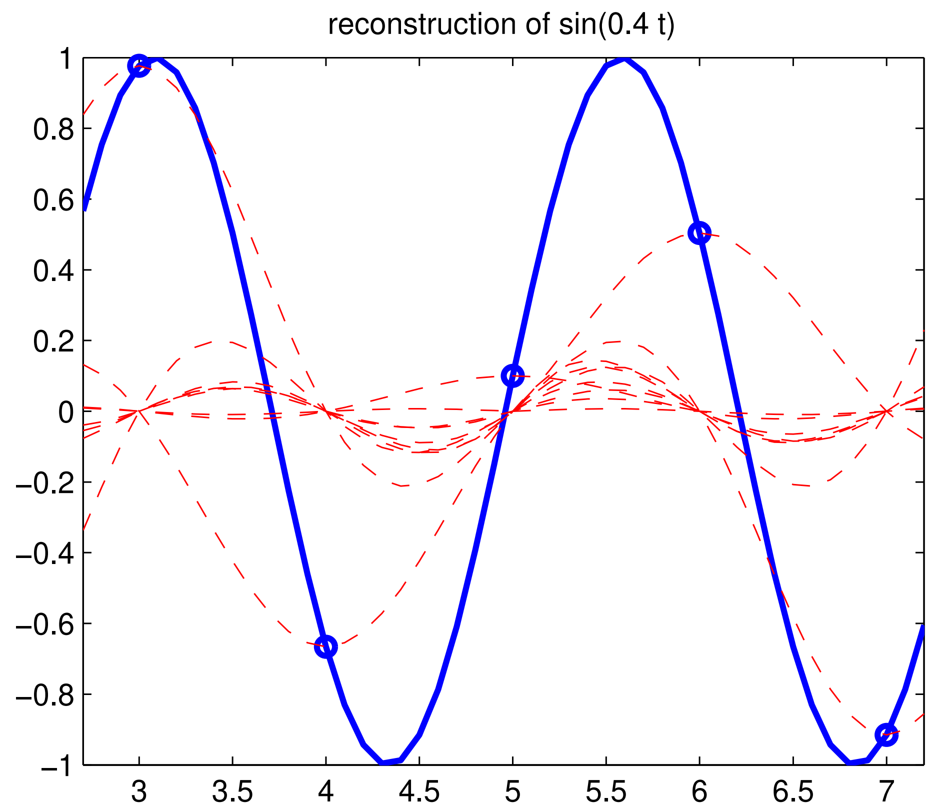
$$x_a(kT_s) = x[k] \; : \qquad \text{interpolation of the given samples}$$

We can interprete $x_a(t)$ as a sum of weighted and shifted impulse responses $h_r(t)$:

$$x_a(t) = \sum_n x[n] h_r(t - nT_s) = \cdots + x[0] h_r(t) + x[1] h_r(t - T_s) + \cdots$$

## Reconstruction (interpretation in time domain)




The sinc-functions take care of the interpolation. Note the locations of the zero crossings of the shifted sinc-functions.

### Reconstruction (interpretation in time domain)

Digital-to-Analog Conversion (DAC) using ideal low-pass filters cannot be implemented:

- $h_r(t)$ is not causal,
- $h_r(t)$ has an impulse response of infinite duration,
- We cannot convert samples $x[n]$ into analog impulses.

In practice, approximations are used (low-pass filters). We will return to this issue in EE2S31 Signal Processing.